Mining Social-Network Graphs

Lecture 8

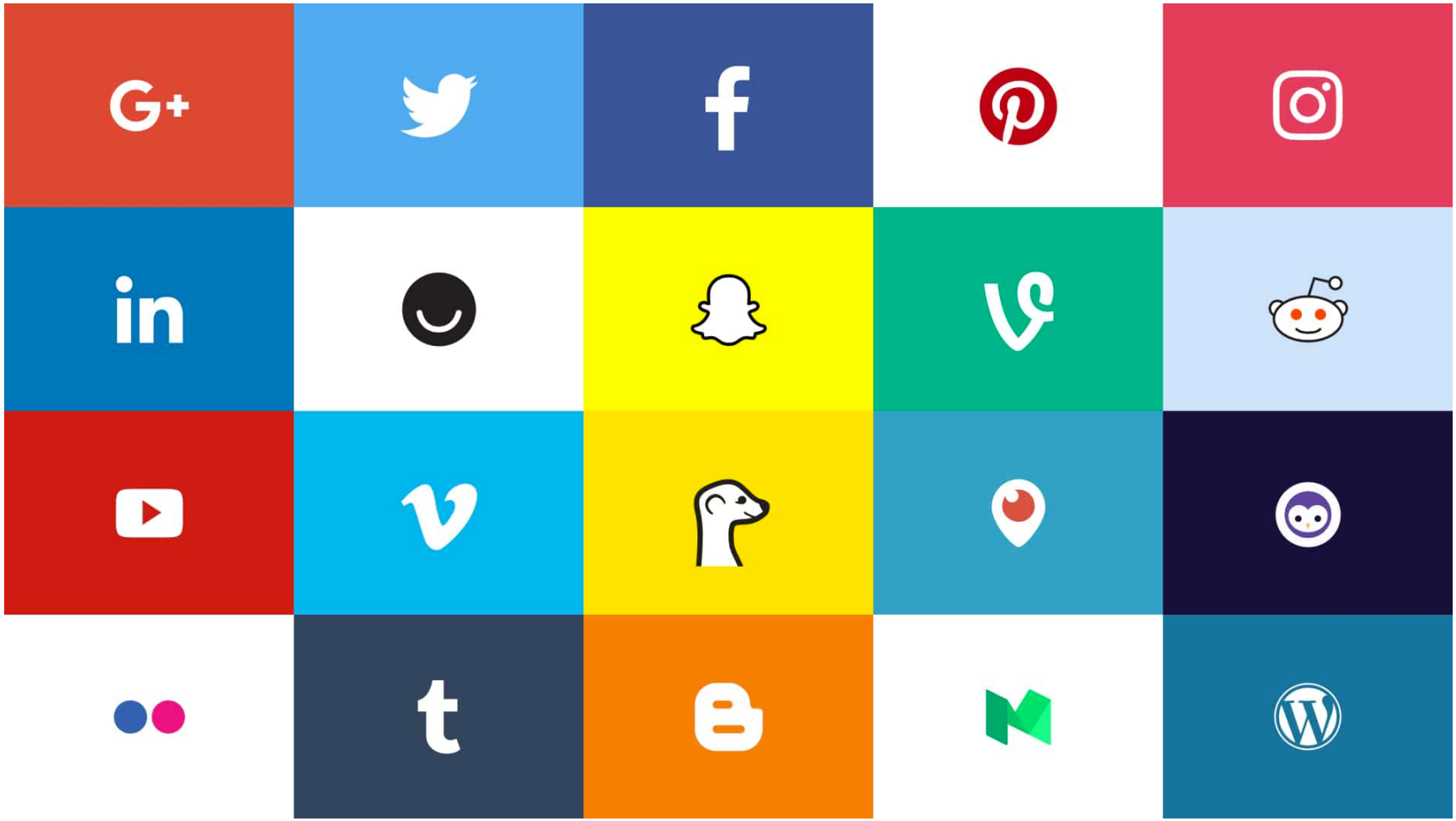
Outline

1. Social Networks as Graphs
   a) Examples
   b) Representation
   c) Properties

2. Clustering
   a) Distance Measures
   b) Girvan-Newman

3. Spectral Analysis
   a) Networks as Matrices
   b) Connectivity
   c) Partitioning
   d) Clustering

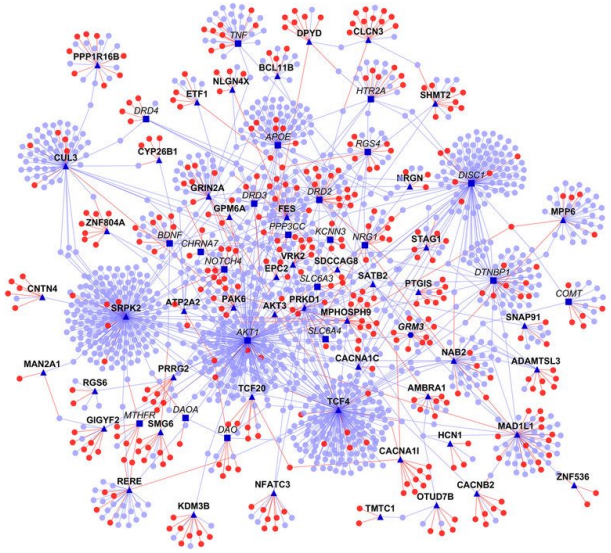
Social Networks (obvious)

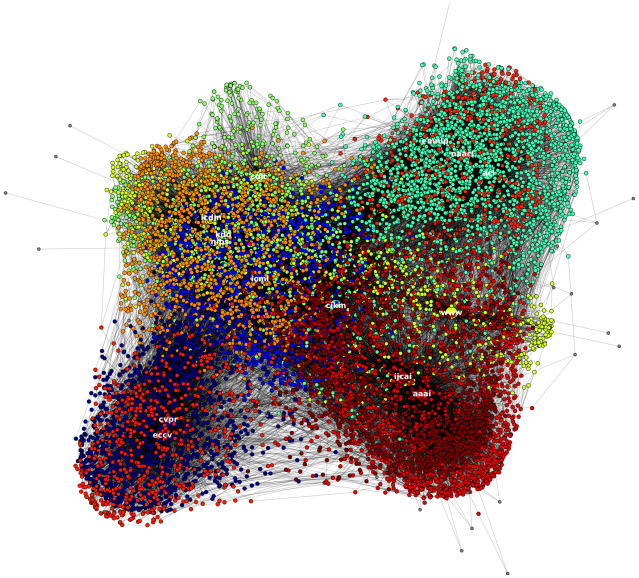


Social Networks (obvious)

Less Obvious

- Phone/E-mail (within K units of time)
- Collaboration (academic papers, patents)
- Biological (proteins, genes)





Representation

- Nodes = entities
  - People, papers, …
  - **Single** vs. multiple
    - del.icio.us: people, websites, tags
- Edges = relationship
  - **Discrete** vs continuous (i.e. weighted)
  - Directed (e.g. following) vs **undirected**

Checkup

- Given a graph of 7 nodes (A-G), how many undirected edges are possible?

Answer

- Given a graph of 7 nodes (A-G), how many undirected edges are possible?

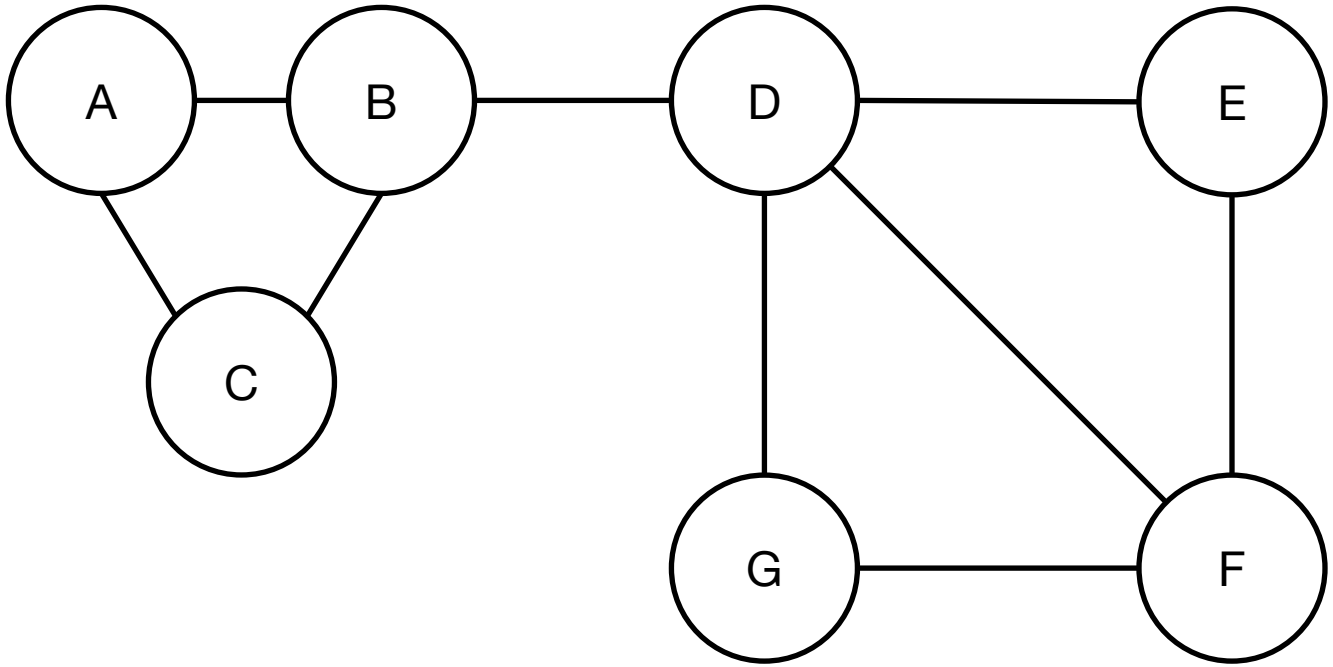
$$\binom{7}{2} = \frac{7!}{2!(7-2)!} = \frac{7(7-1)}{2} = 21$$

Locality

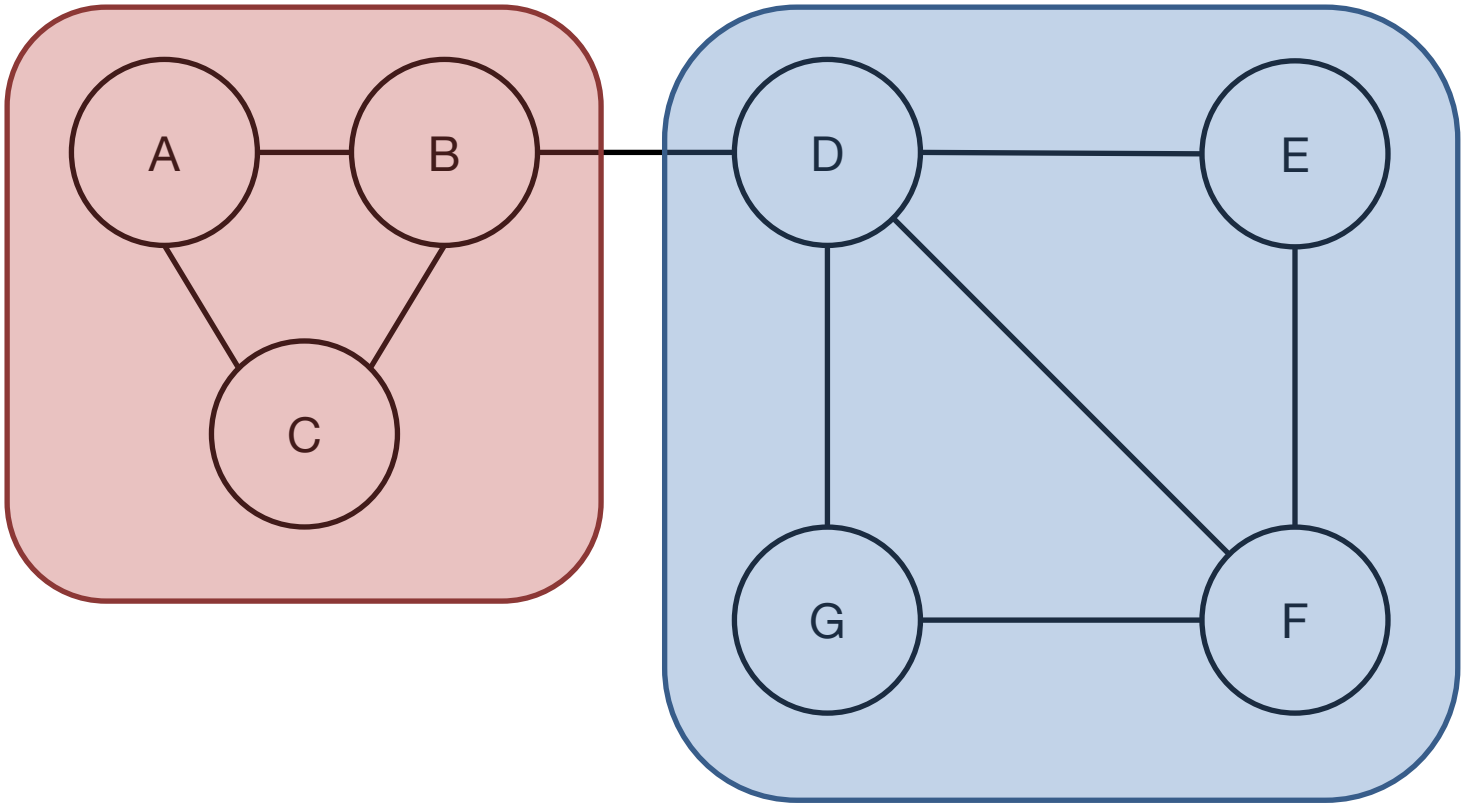
- Typically assumed that if node A is connected to both B and C, then it is more likely than random that B and C are connected

- Our focus: **Community Detection**
  - Finding groups of densely connected nodes
  - (Tightly related to the locality assumption – i.e. would not work well on random graphs)

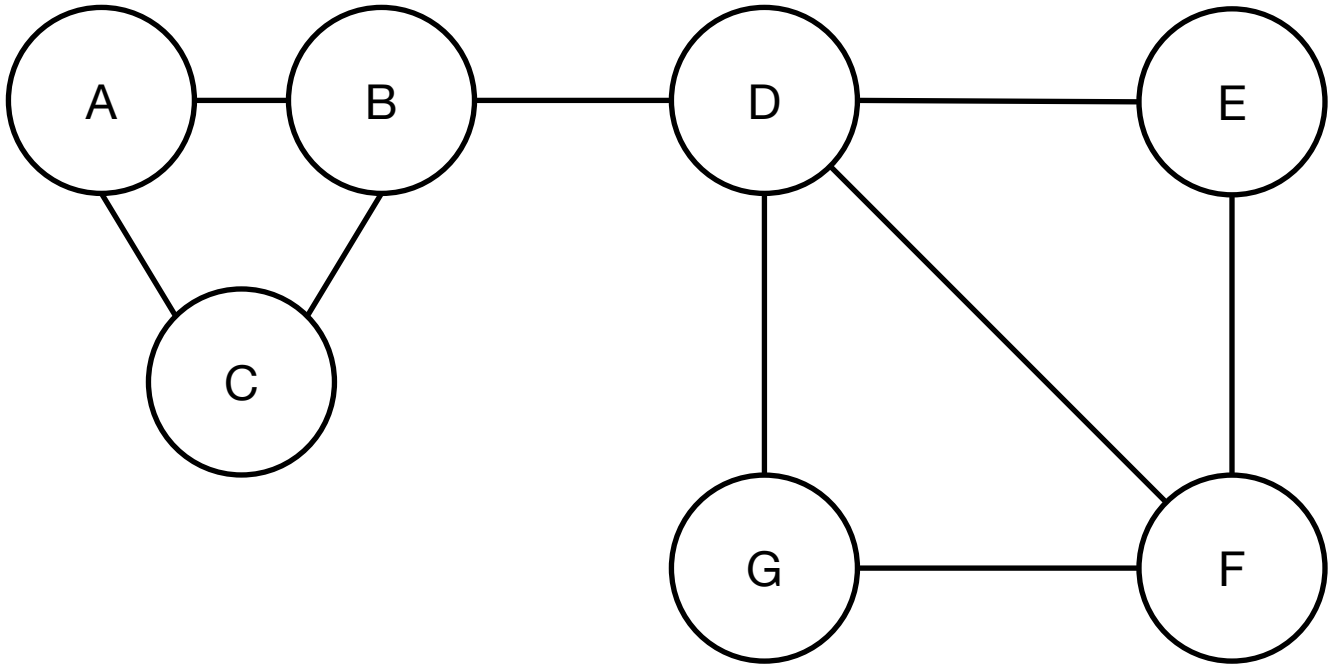
Example Graph



Communities to Be Detected

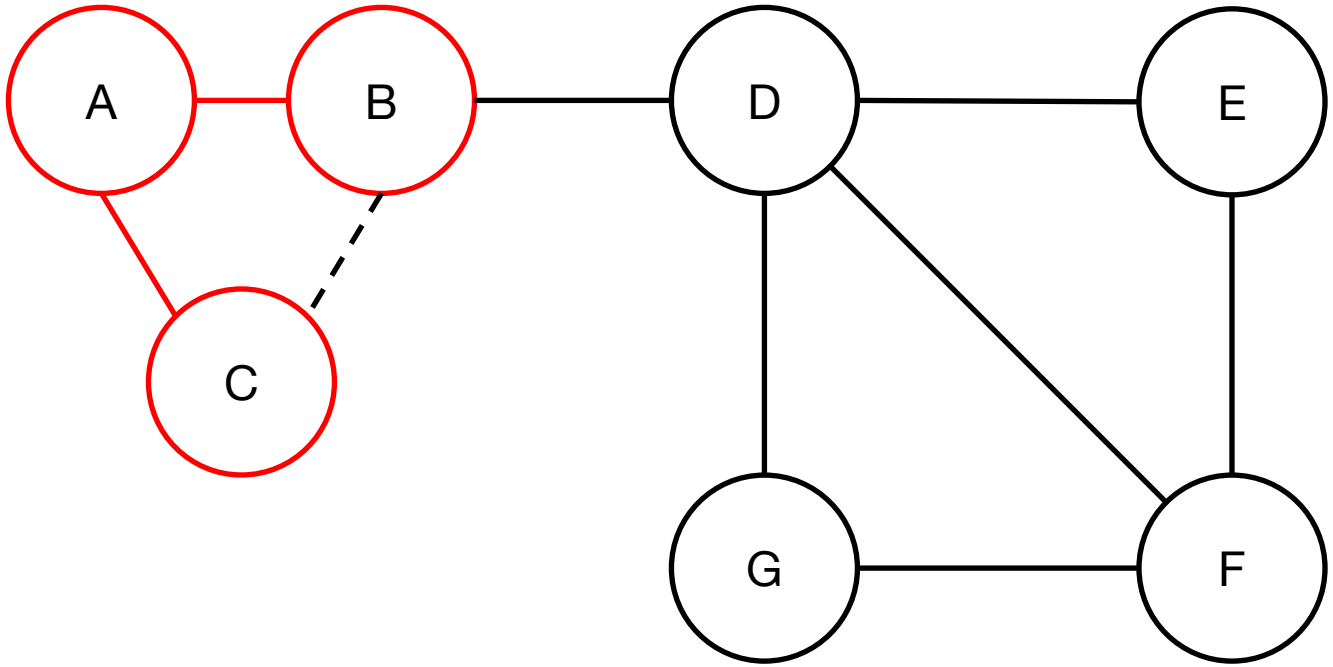


Nodes, Edges, Average Connectivity?



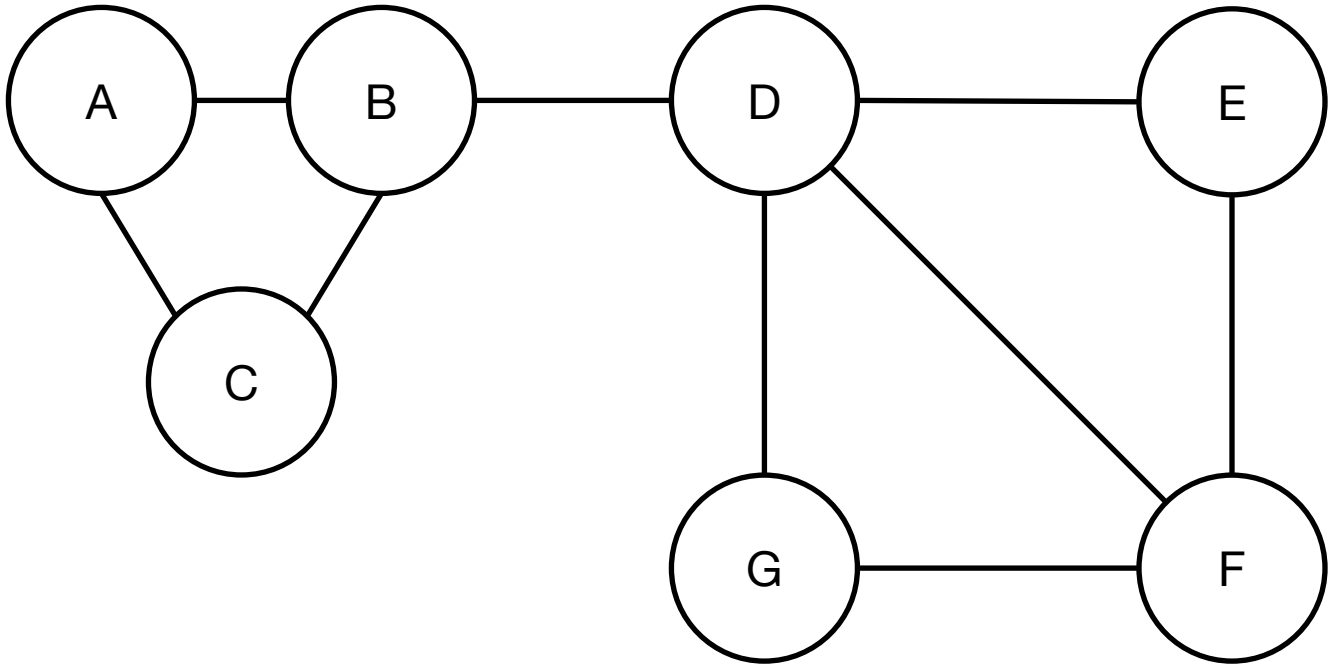
- Nodes: 7
- Edges: 9
- Avg Connectivity: 9/21 ~ 0.43

Given A-B, A-C; Expected B-C?



- Nodes: 7
- Edges: 9
- Avg Connectivity: 9/21 ~ 0.43
- Expected Locality: 7/19 ~ 0.37

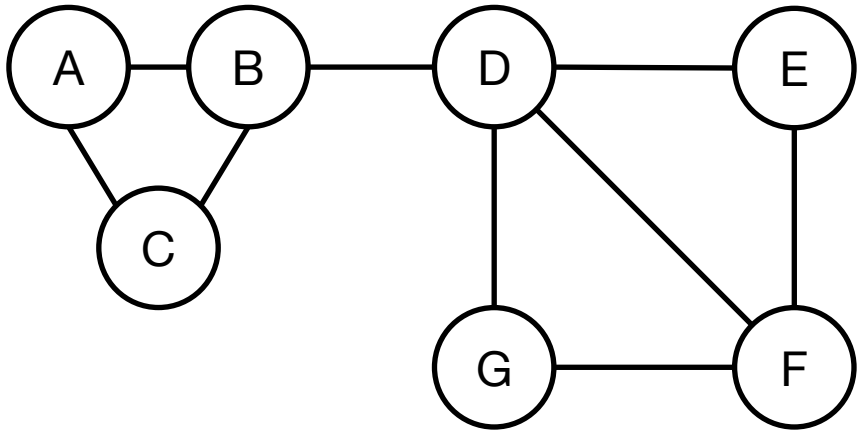
Actual?



- Nodes: 7
- Edges: 9
- Avg Connectivity: 9/21 ~ 0.43
- Expected Locality: 7/19 ~ 0.37
- *What are all the triples?*

Evaluate Node Triples

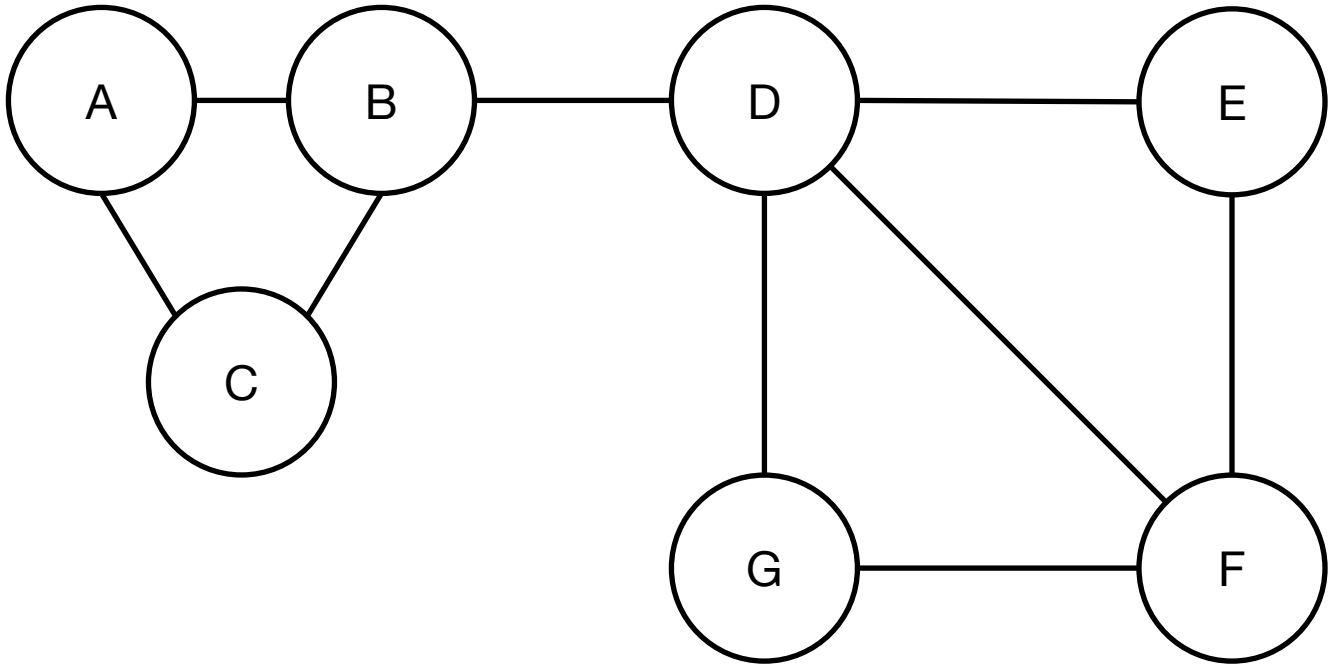
- **A**
  - BC ✅
- **B**
  - AC ✅, AD ❌
  - CD ❌
- **C**
  - AB ✅
- **D**
  - BE ❌, BF ❌, BG ❌
  - EF ✅, EG ❌
  - FG ✅
- **E**
  - DF ✅



- **F**
  - DE ✅, DG ✅
  - EG ❌
- **G**
  - DF ✅

- ✅ = 9, ❌ = 7
- ✅ / (✅ + ❌) = 9/16 ~ 0.56

Local!

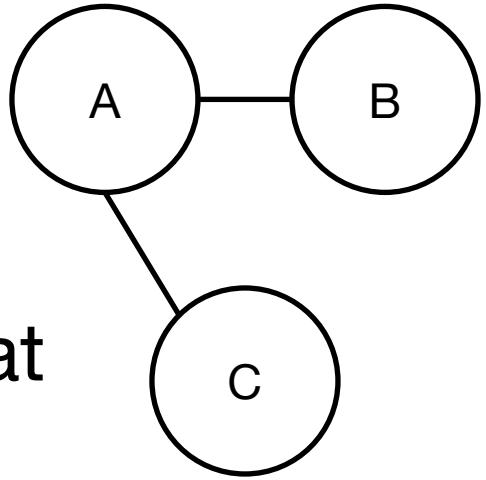


- Nodes: 7
- Edges: 9
- Avg Connectivity: 9/21 ~ 0.43
- Expected Locality: 7/19 ~ 0.37
- Actual Locality: 9/16 ~ 0.56 (**>> Expected!**)

Let's Cluster!

Distance Measure?

- Distance measures on social-network graphs can be tricky
- KISS ex: D(x,y)=0 if edge, 1 otherwise
  - D(A, B)=0
  - D(A, C)=0
  - D(B, C) = 1 > D(A, B) + D(A, C)



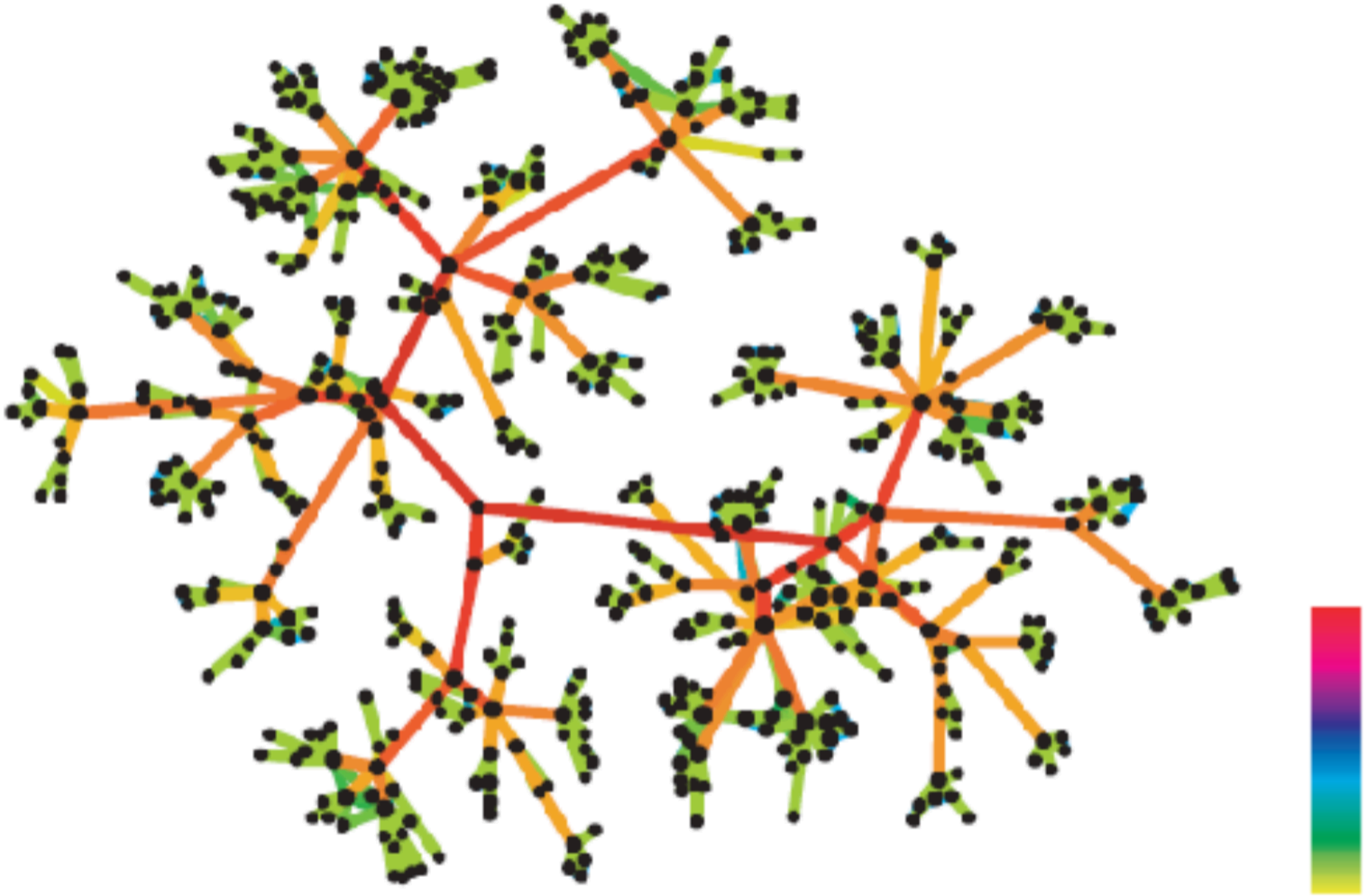
- Also leads to unfortunate ties that lead to unfortunate results in typical clustering algorithms

Betweenness

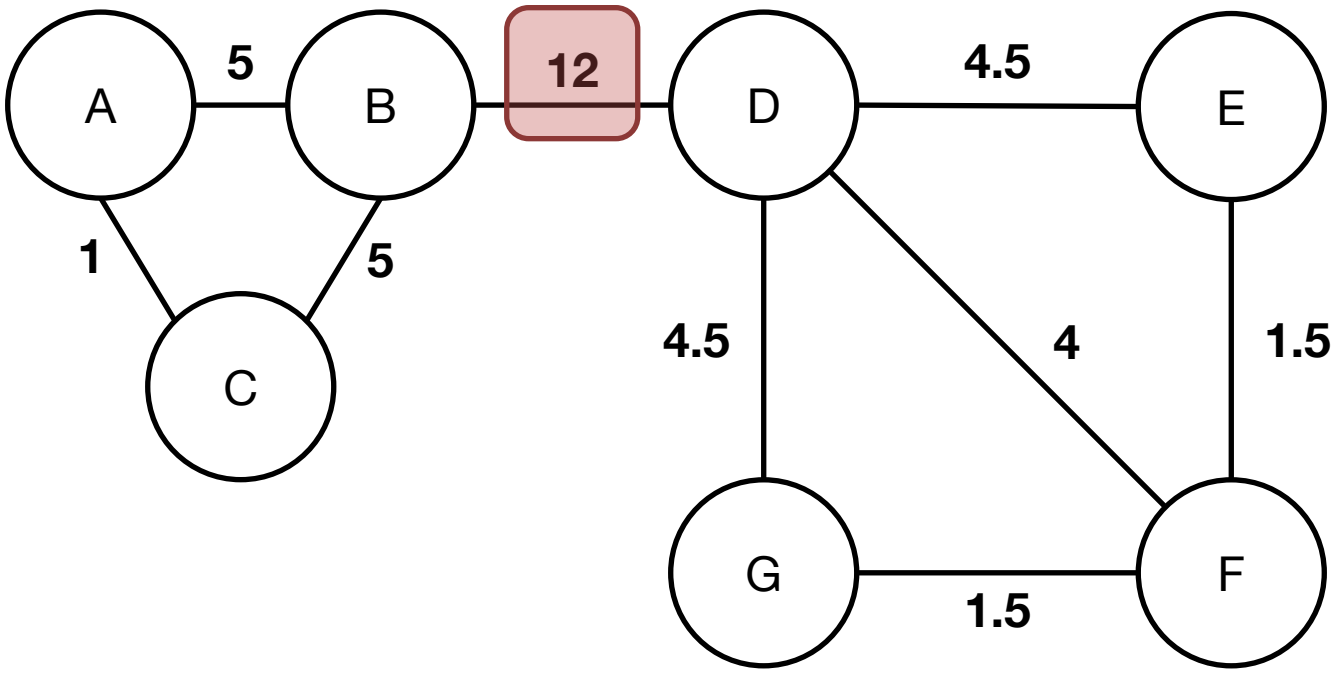
- One of the simplest measures, based on finding the edges that are the *least likely* to be inside a community
  - Clustering: remove high betweenness first!

- B(a, b) = for all pairs of nodes (x, y), fraction of shortest paths that include edge ab

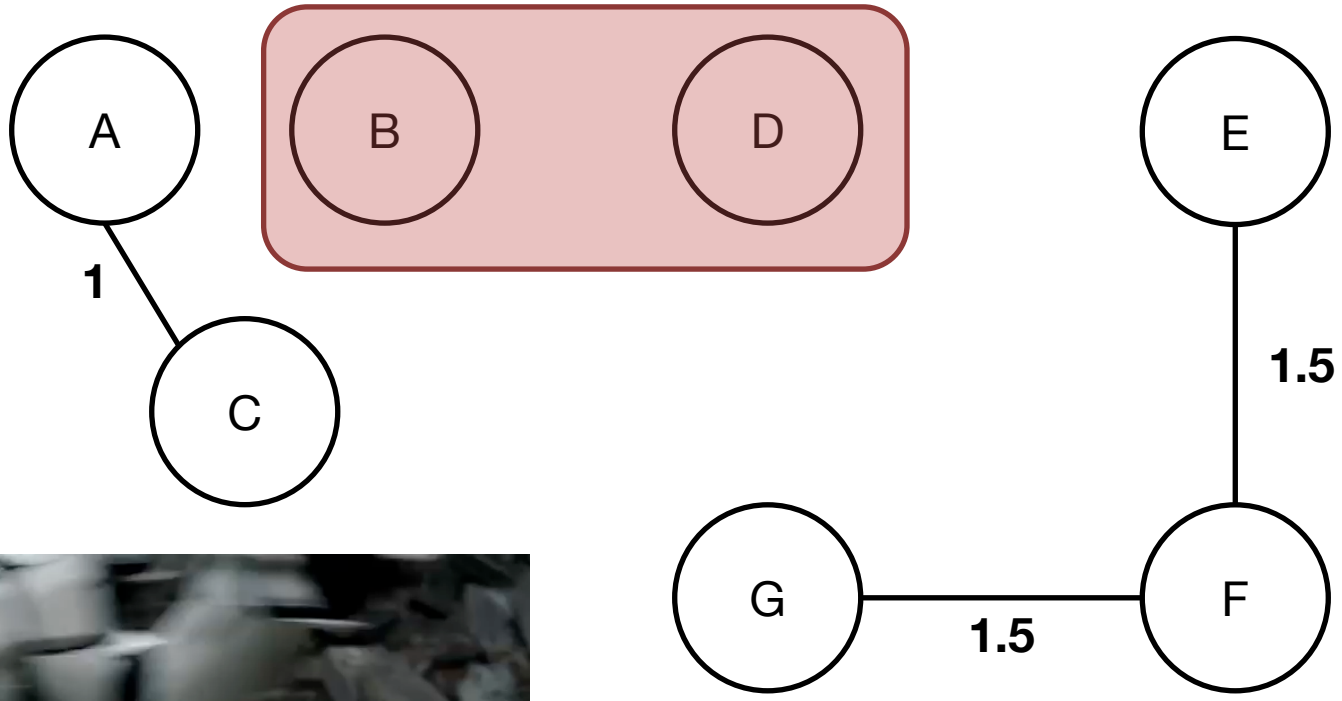
Betweenness Intuition (1)



Betweenness Intuition (2)



Betweenness Intuition (3)

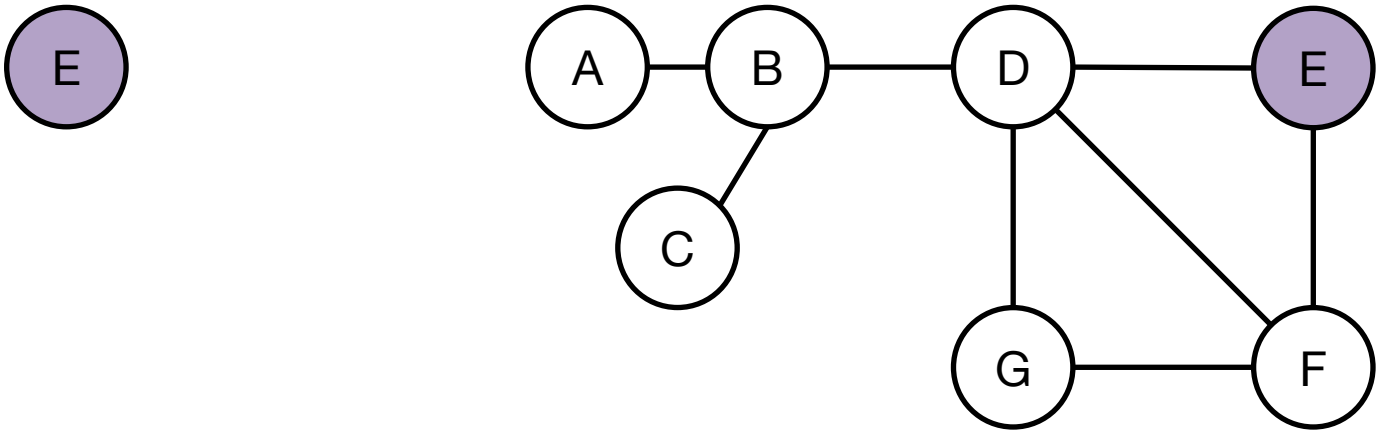


Computing Graph Betweenness

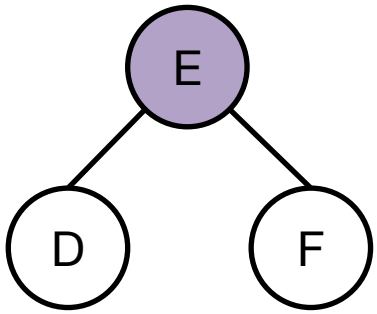
1. For each node
   a) Breadth-First Search (BFS)
      - WHY?
   b) For each edge, compute betweenness

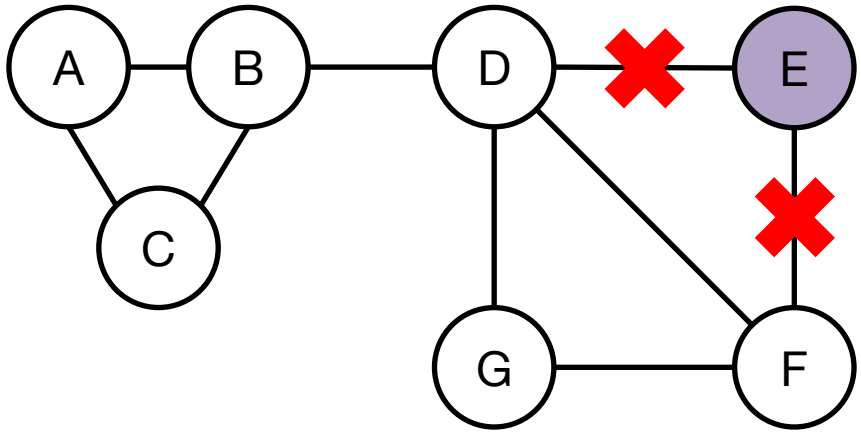
2. For each edge
   a) Sum contributions from trees above
   b) Divide by 2 (prevents duplication)

Example (E, BFS.0)

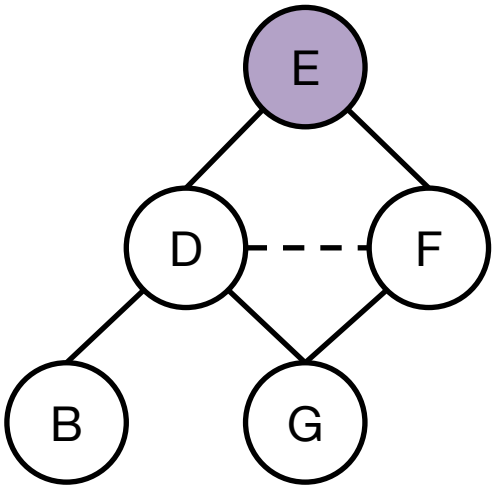


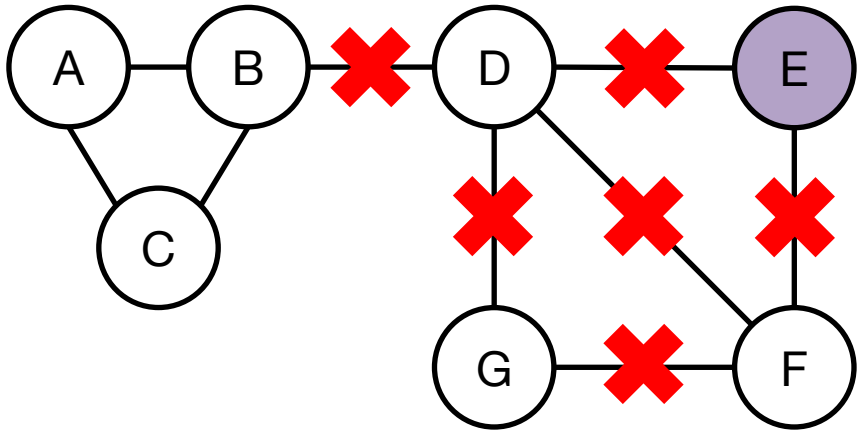
Example (E, BFS.1)



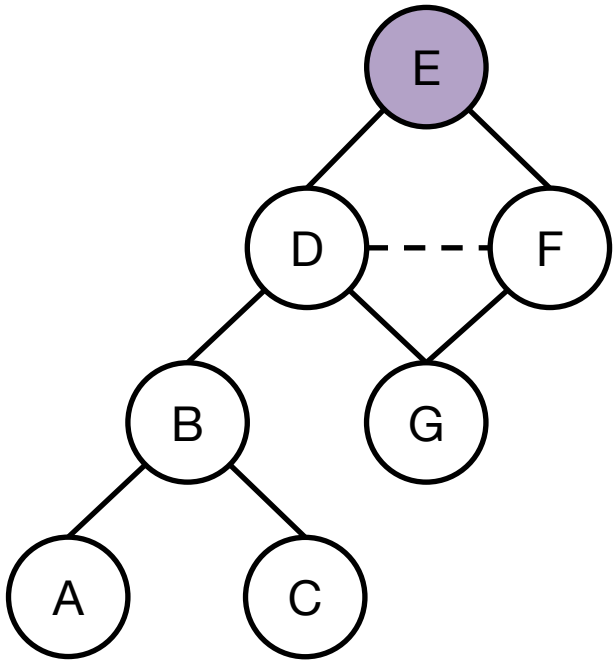


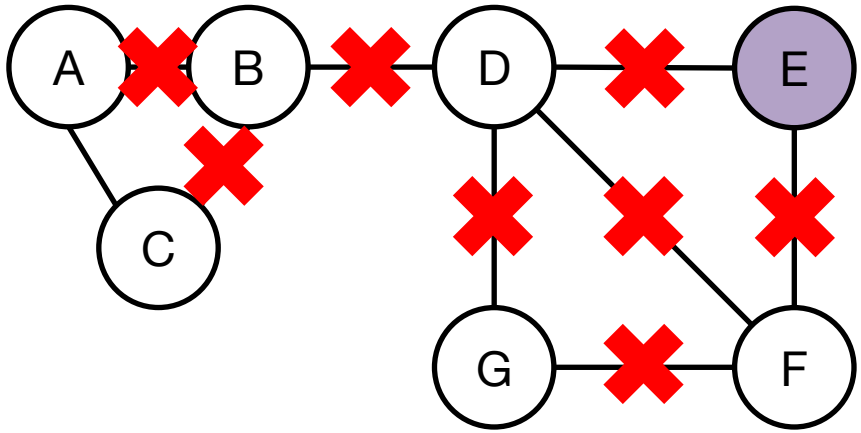
Example (E, BFS.2)



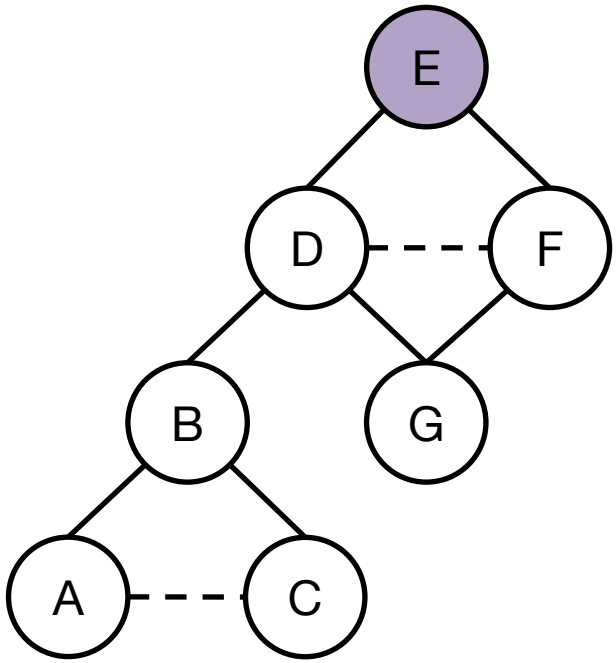


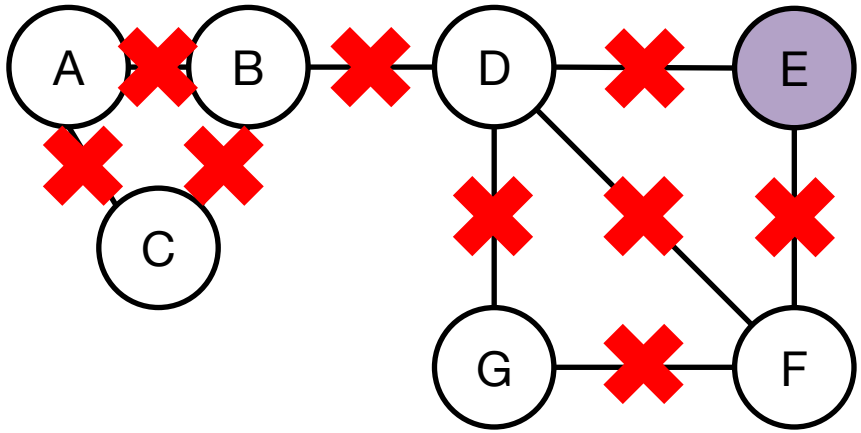
Example (E, BFS.3)





Example (E, BFS.Done)



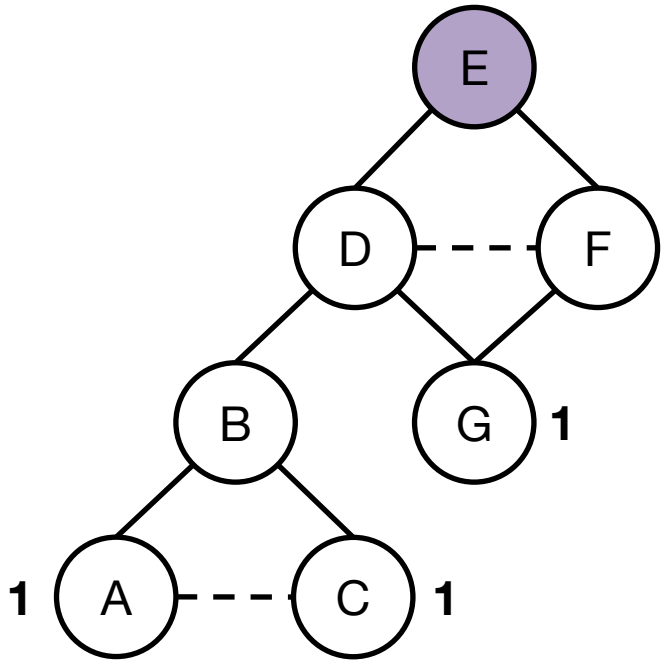


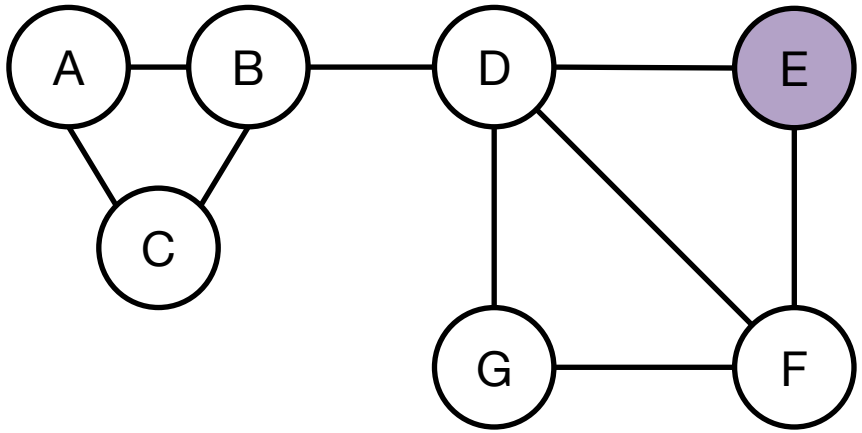
Computing BFS Betweenness

- Recursive credit definition
  - Node gets 1 + sum of edge credits below
  - Edge gets node below divided by outgoing
    - Ignore between nodes at the same level (never used for shortest paths)

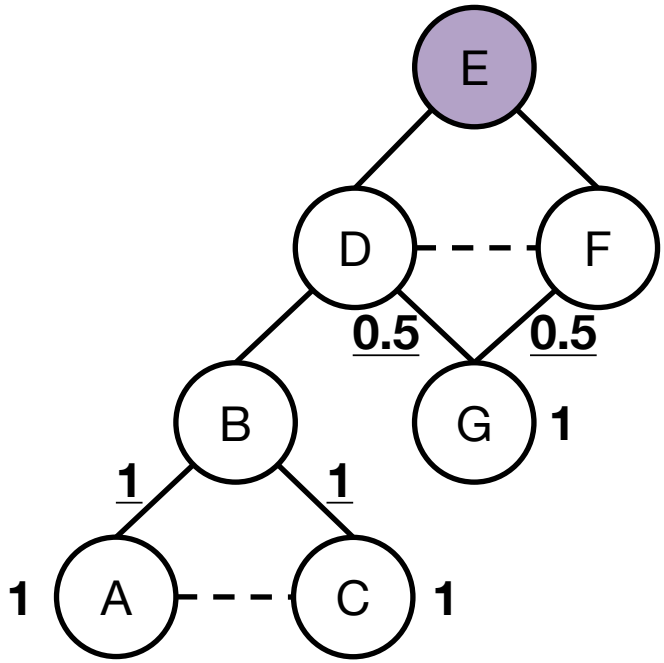
- Start from leaves, move up

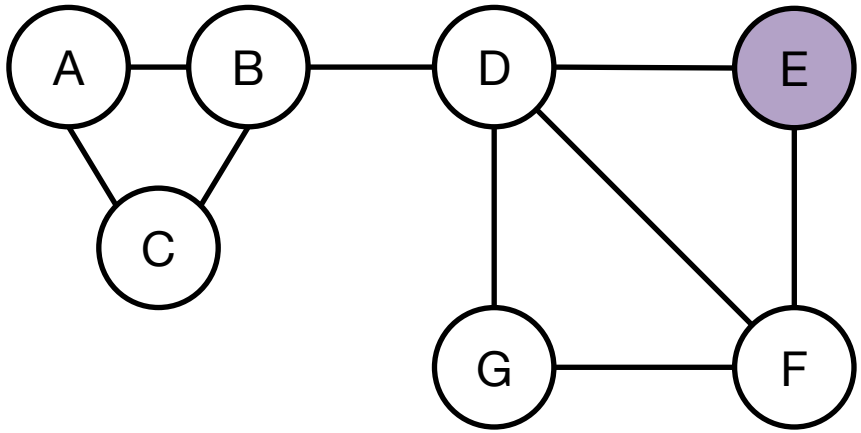
Example (E, Credits.1)



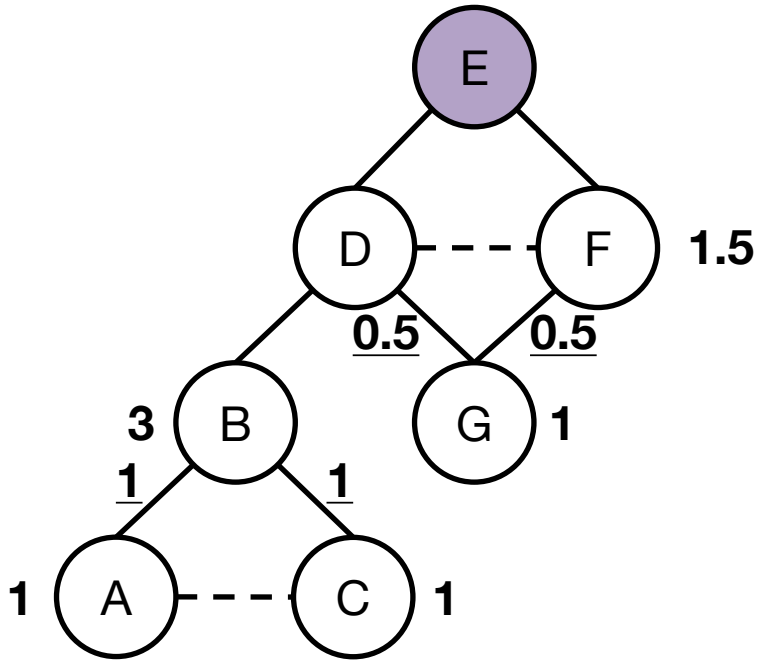


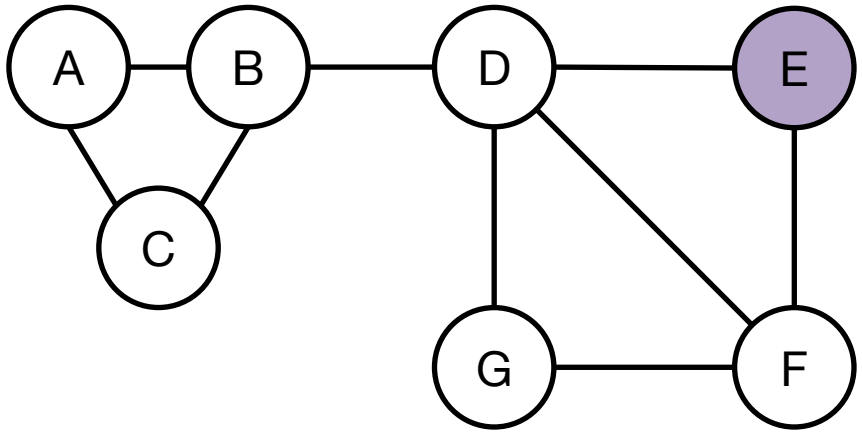
Example (E, Credits.2)



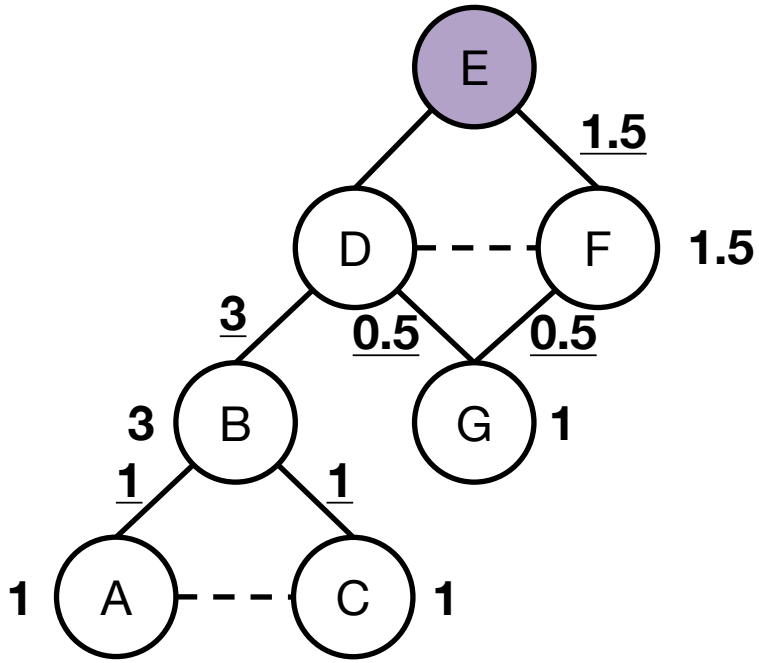


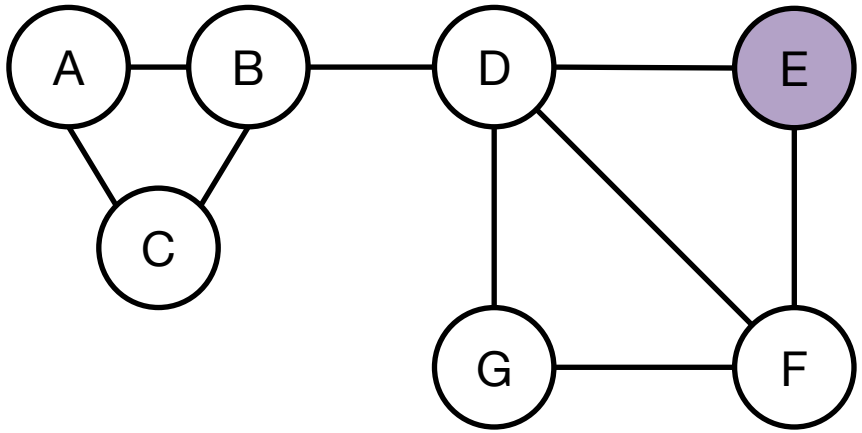
Example (E, Credits.3)



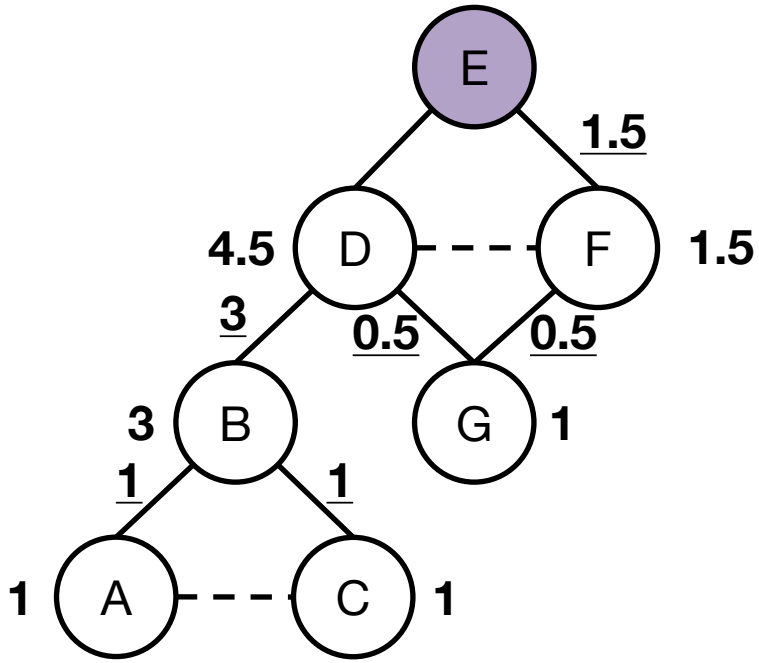


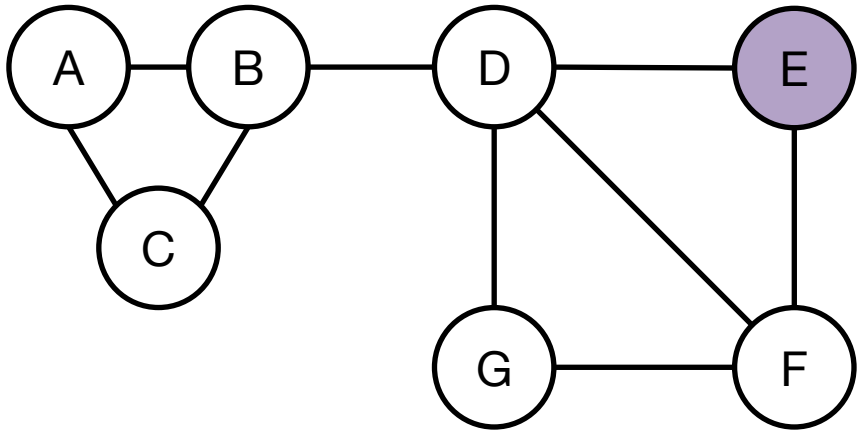
Example (E, Credits.4)



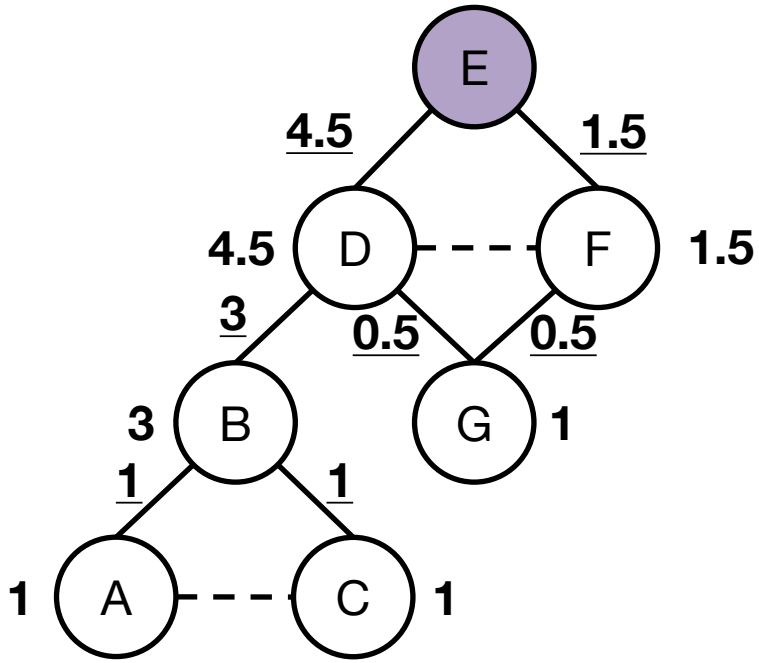


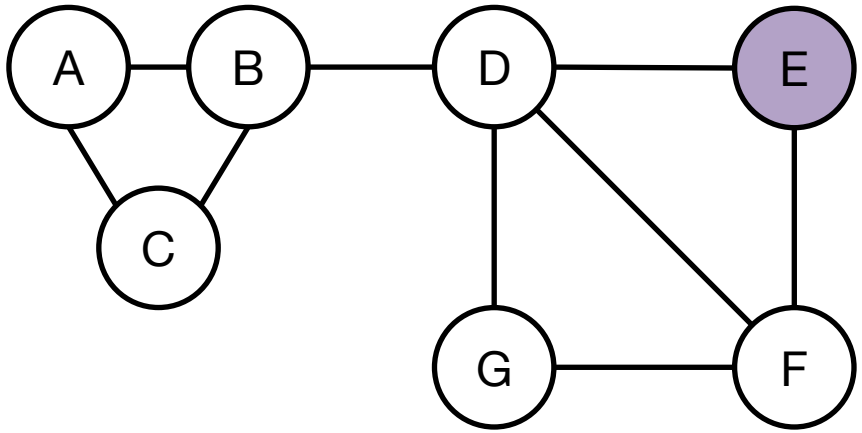
Example (E, Credits.5)



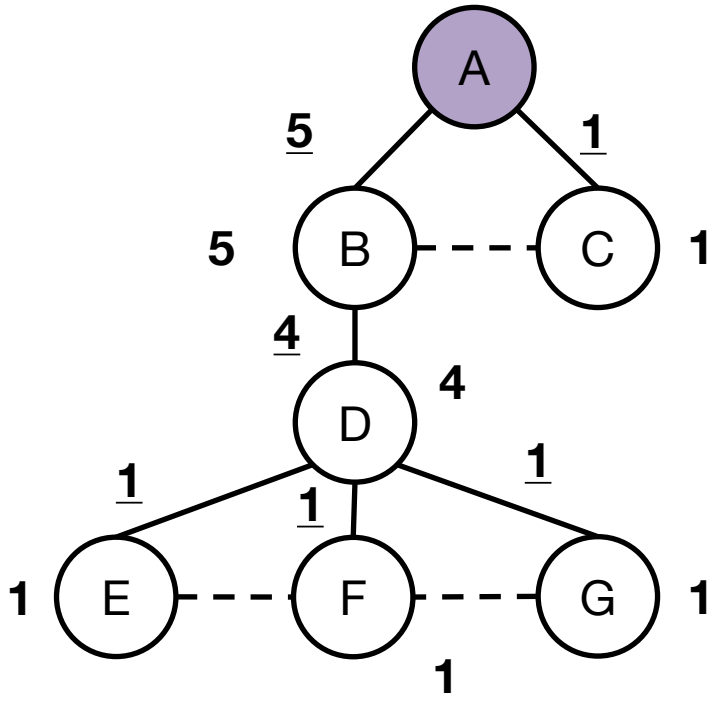


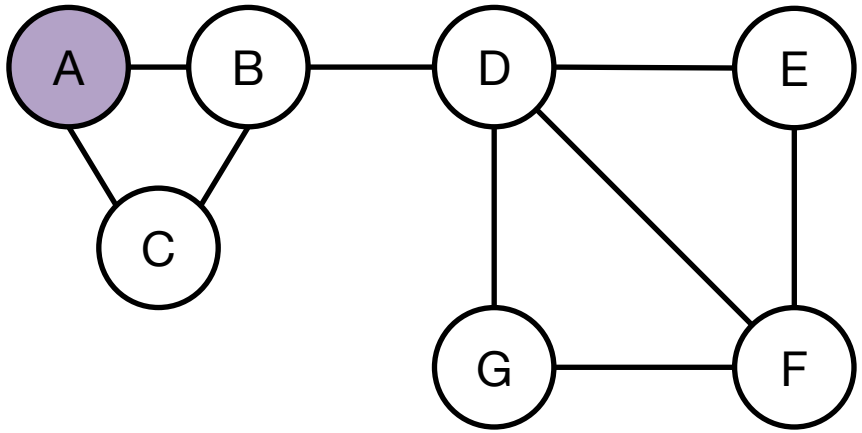
Example (E, Credits.6)



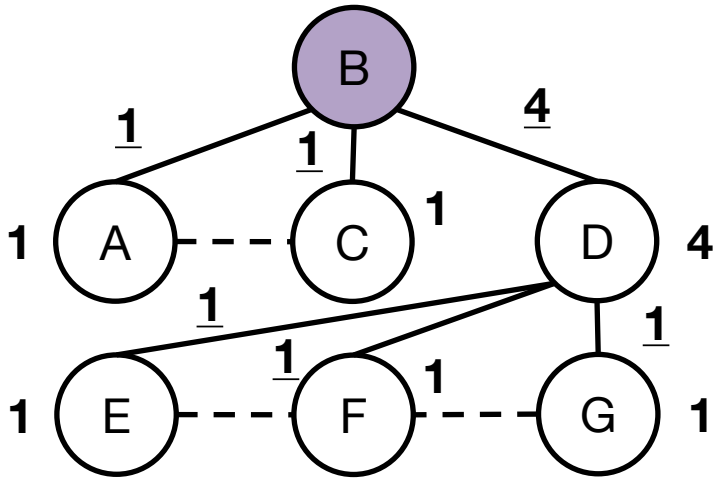


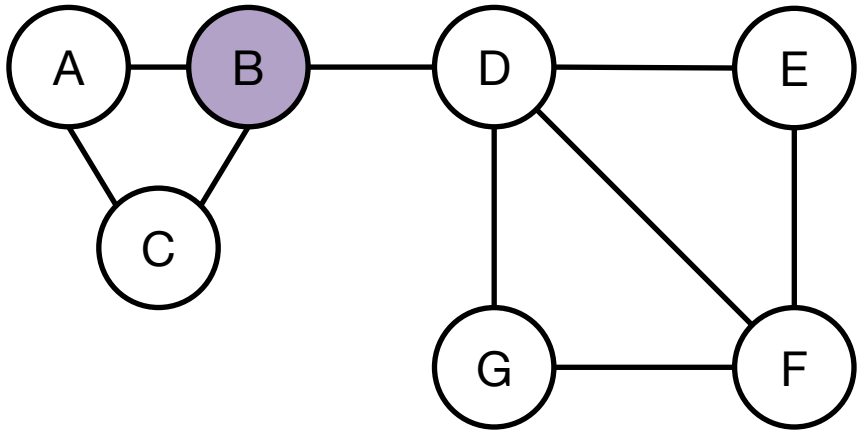
Example (A, Credits)



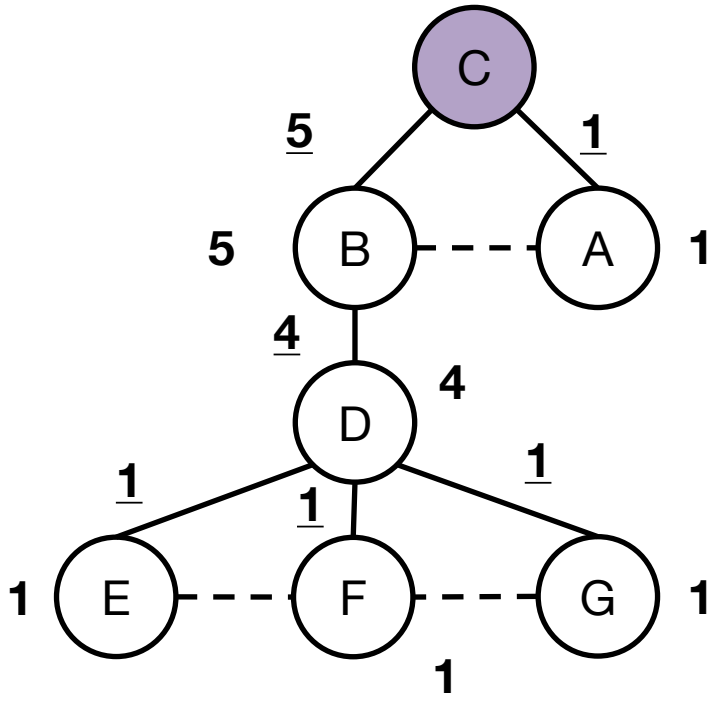


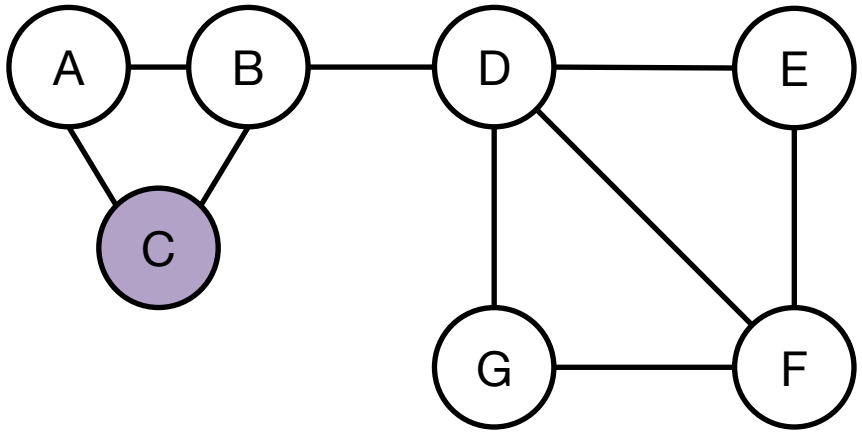
Example (B, Credits)



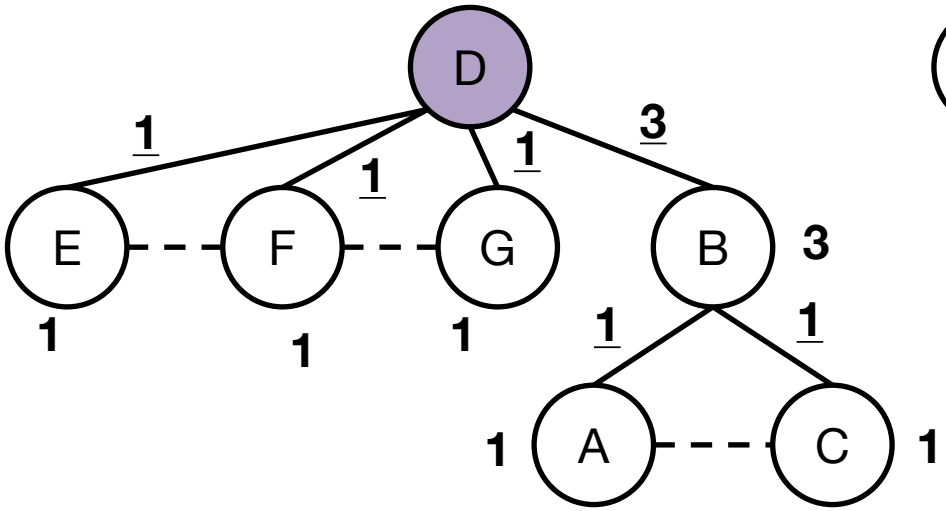


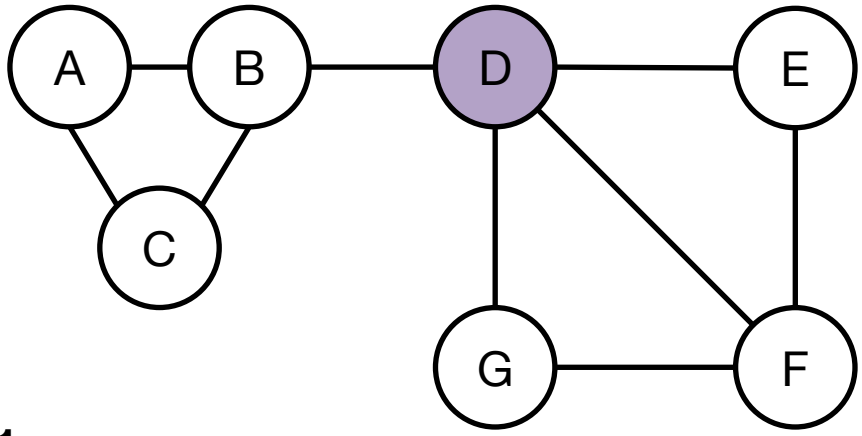
Example (C, Credits)



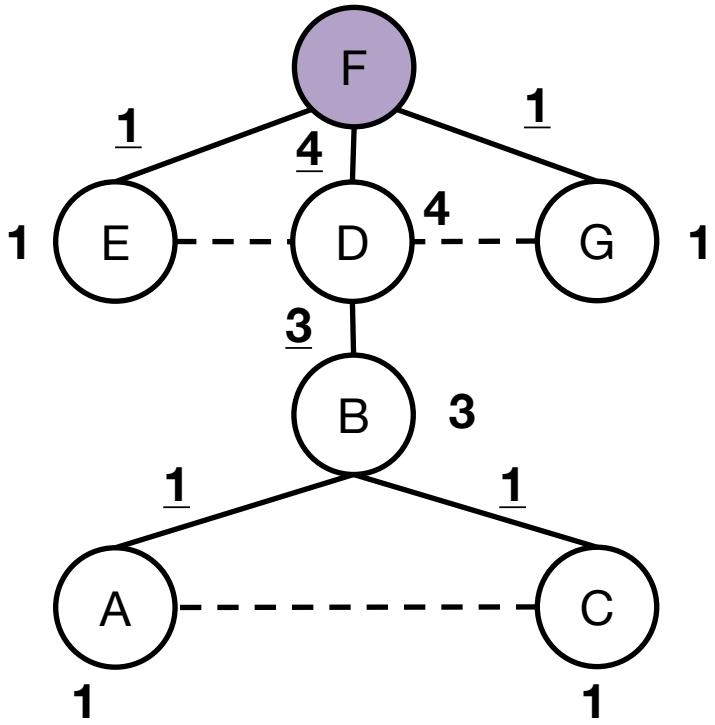


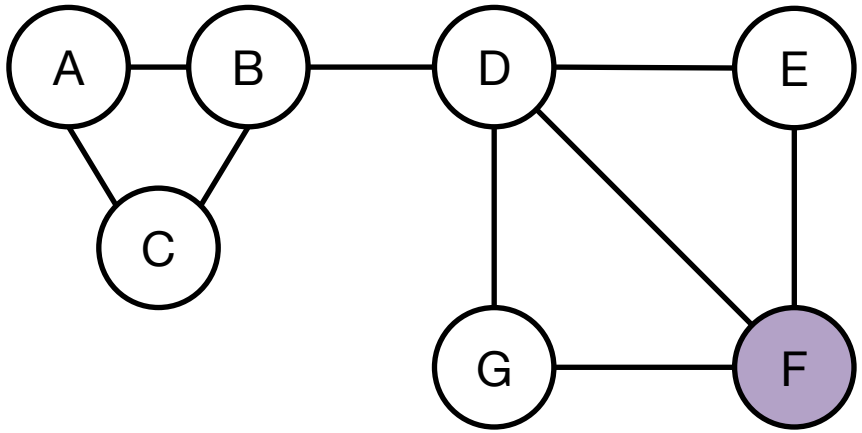
Example (D, Credits)



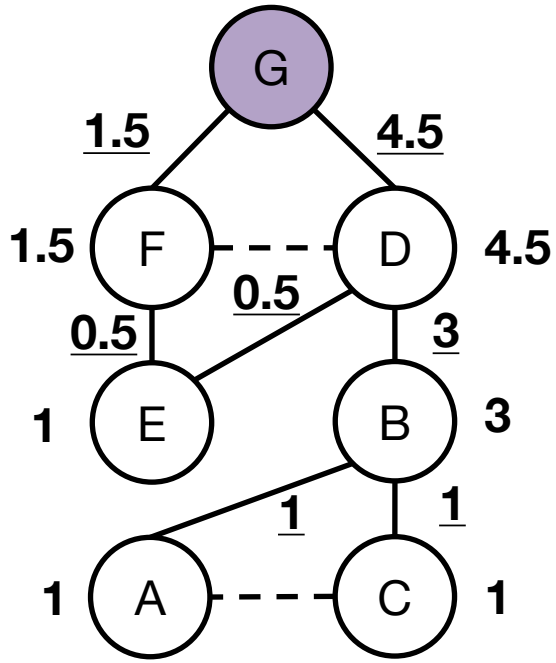


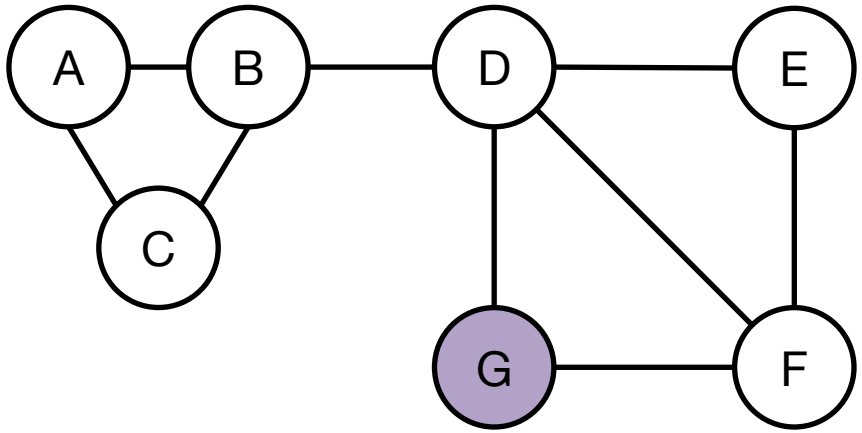
Example (F, Credits)





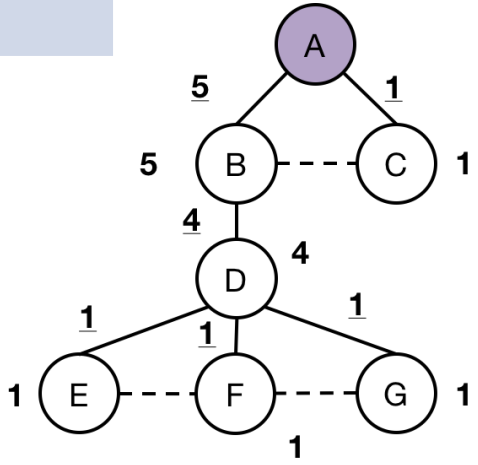
Example (G, Credits)





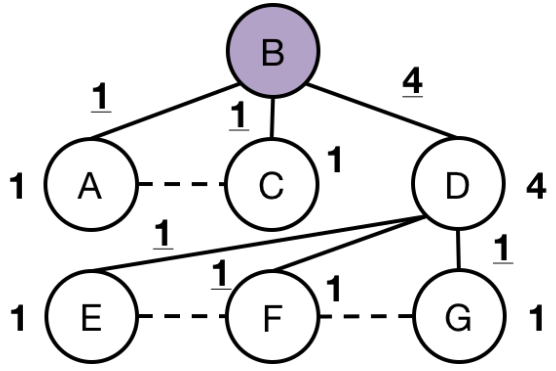
Sum Contributions (A)

	AB	AC	BC	BD	DE	DG	DF	EF	GF
A	5	1		4	1	1	1		
B									
C									
D									
E									
F									
G									



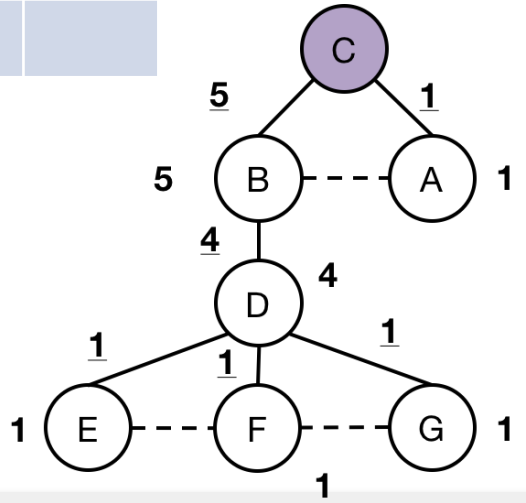
Sum Contributions (B)

	AB	AC	BC	BD	DE	DG	DF	EF	GF
A	5	1		4	1	1	1		
B	1		1	4	1	1	1		
C									
D									
E									
F									
G									



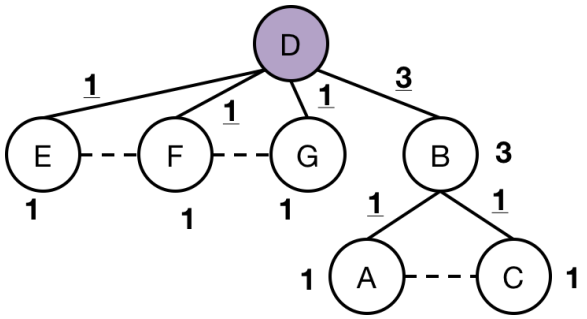
Sum Contributions (C)

	AB	AC	BC	BD	DE	DG	DF	EF	GF
A	5	1		4	1	1	1		
B	1		1	4	1	1	1		
C		1	5	4	1	1	1		
D									
E									
F									
G									



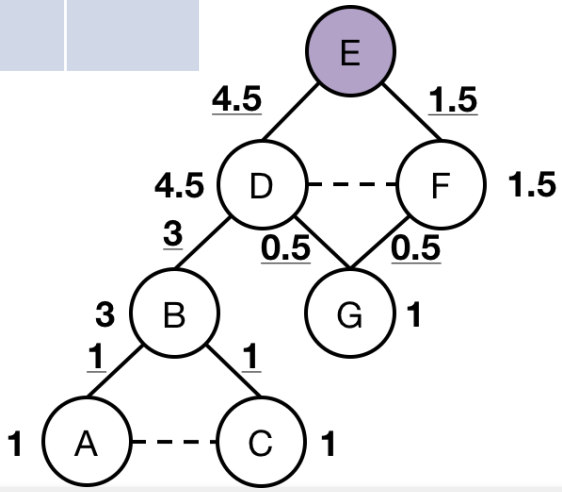
Sum Contributions (D)

	AB	AC	BC	BD	DE	DG	DF	EF	GF
A	5	1		4	1	1	1		
B	1		1	4	1	1	1		
C		1	5	4	1	1	1		
D	1		1	3	1	1	1		
E									
F									
G									



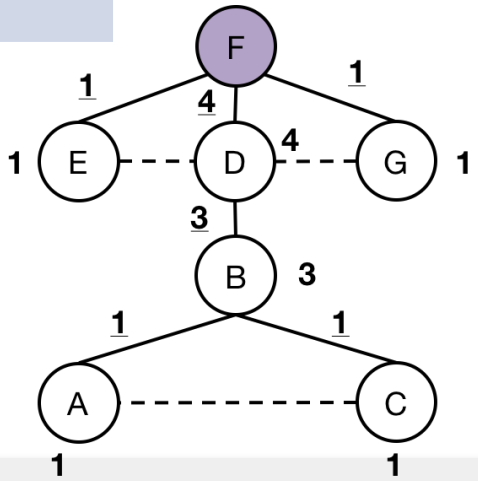
Sum Contributions (E)

	AB	AC	BC	BD	DE	DG	DF	EF	GF
A	5	1		4	1	1	1		
B	1		1	4	1	1	1		
C		1	5	4	1	1	1		
D	1		1	3	1	1	1		
E	1		1	3	4.5	0.5		1.5	0.5
F									
G									



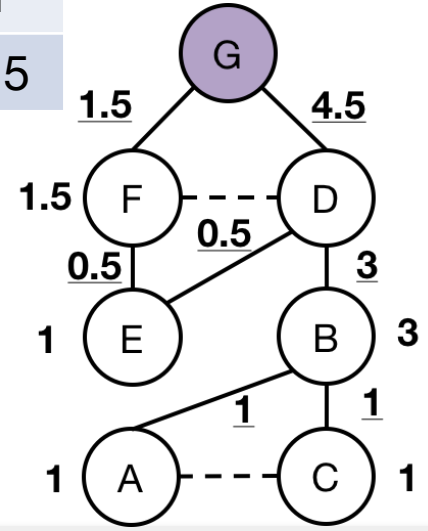
Sum Contributions (F)

	AB	AC	BC	BD	DE	DG	DF	EF	GF
A	5	1		4	1	1	1		
B	1		1	4	1	1	1		
C		1	5	4	1	1	1		
D	1		1	3	1	1	1		
E	1		1	3	4.5	0.5		1.5	0.5
F	1		1	3			4	1	1
G									



Sum Contributions (G)

	AB	AC	BC	BD	DE	DG	DF	EF	GF
A	5	1		4	1	1	1		
B	1		1	4	1	1	1		
C		1	5	4	1	1	1		
D	1		1	3	1	1	1		
E	1		1	3	4.5	0.5		1.5	0.5
F	1		1	3			4	1	1
G	1		1	3	0.5	4.5		0.5	1.5



Sum Contributions (+)

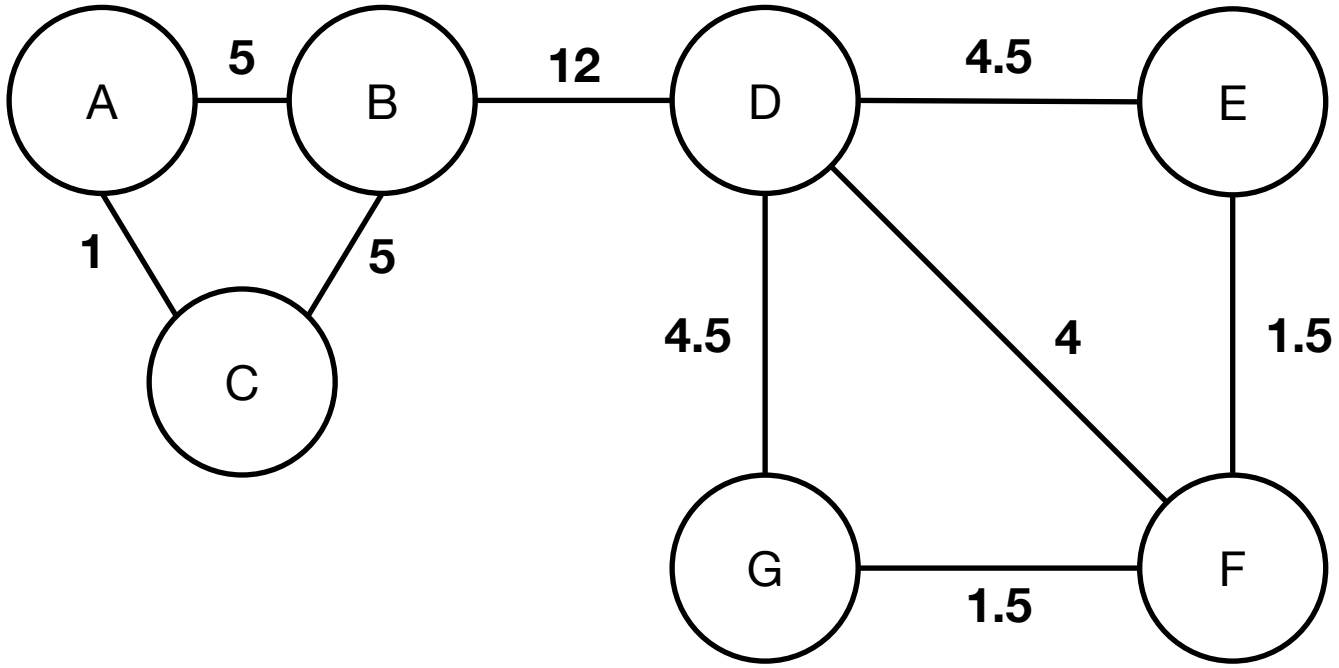
	AB	AC	BC	BD	DE	DG	DF	EF	GF
A	5	1		4	1	1	1		
B	1		1	4	1	1	1		
C		1	5	4	1	1	1		
D	1		1	3	1	1	1		
E	1		1	3	4.5	0.5		1.5	0.5
F	1		1	3			4	1	1
G	1		1	3	0.5	4.5		0.5	1.5
+	10	2	10	24	9	9	8	3	3

Sum Contributions (/2)

	AB	AC	BC	BD	DE	DG	DF	EF	GF
A	5	1		4	1	1	1		
B	1		1	4	1	1	1		
C		1	5	4	1	1	1		
D	1		1	3	1	1	1		
E	1		1	3	4.5	0.5		1.5	0.5
F	1		1	3			4	1	1
G	1		1	3	0.5	4.5		0.5	1.5
+	10	2	10	24	9	9	8	3	3
/2	5	1	5	12	4.5	4.5	4	1.5	1.5

Edge Contributions

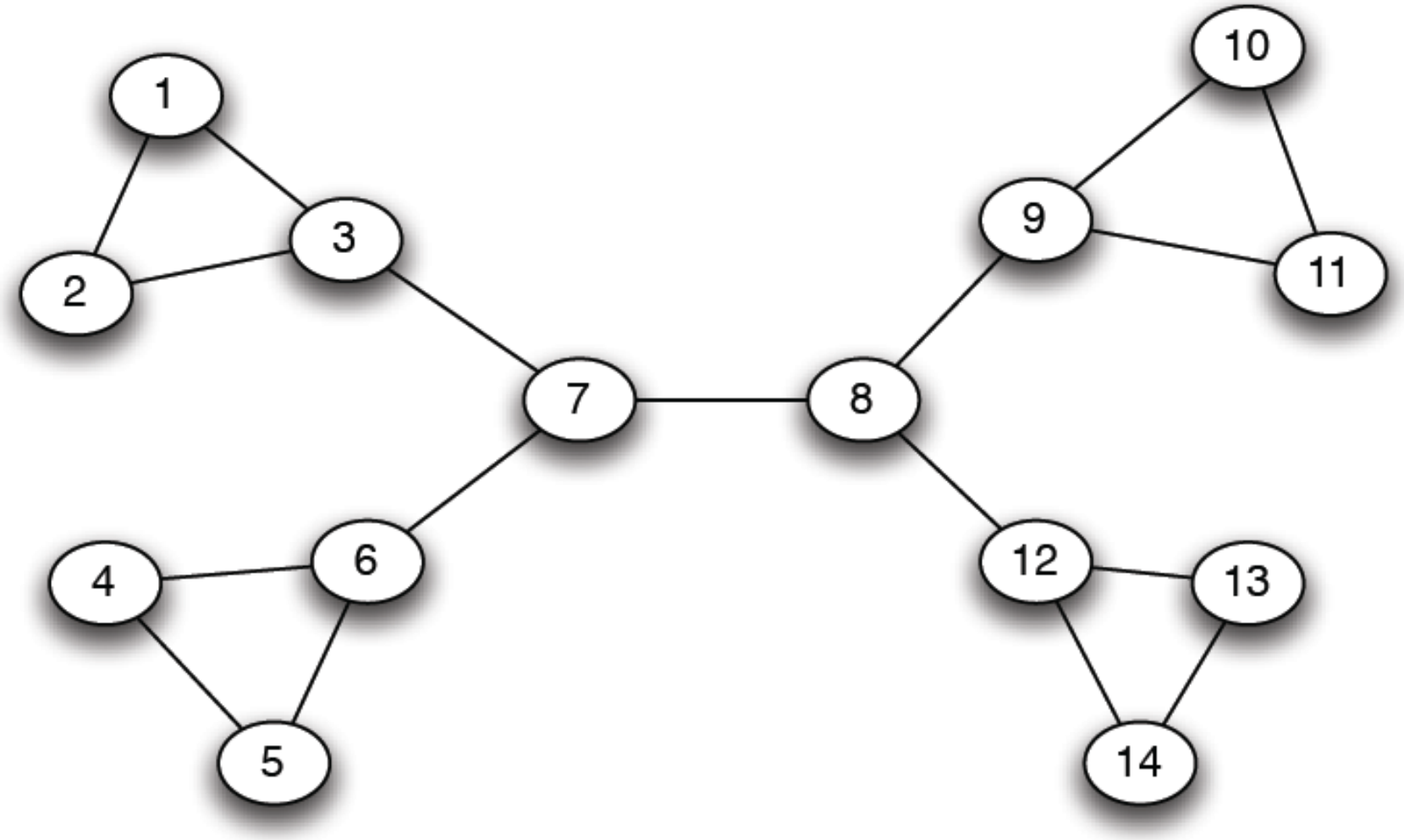
AB	AC	BC	BD	DE	DG	DF	EF	GF
5	1	5	12	4.5	4.5	4	1.5	1.5



Girvan-Newman

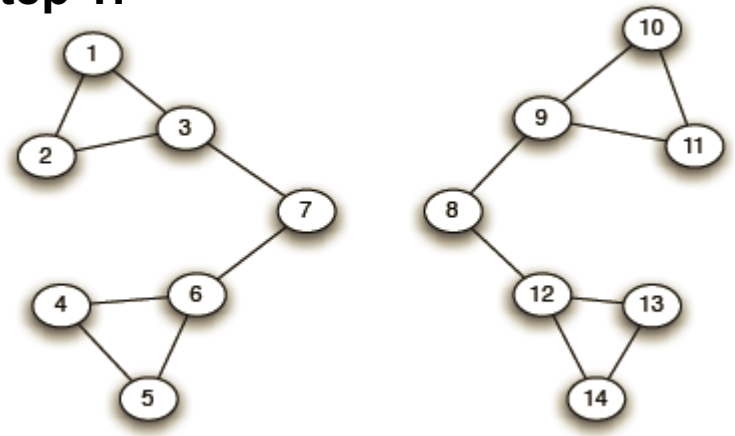
1. Repeat until no edges left
   a) Calculate betweenness of edges
   b) Remove edge(s) with highest betweenness

Example (0)

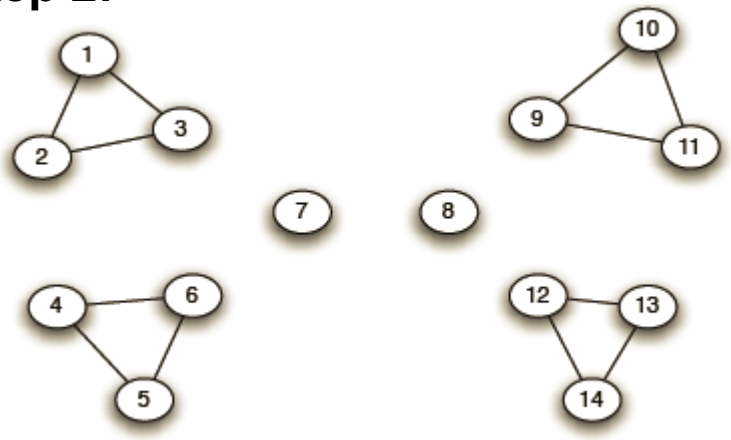


Example (1)

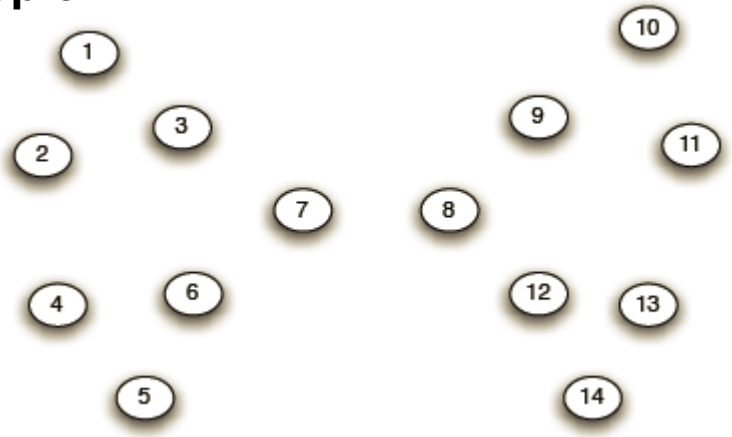
Step 1:



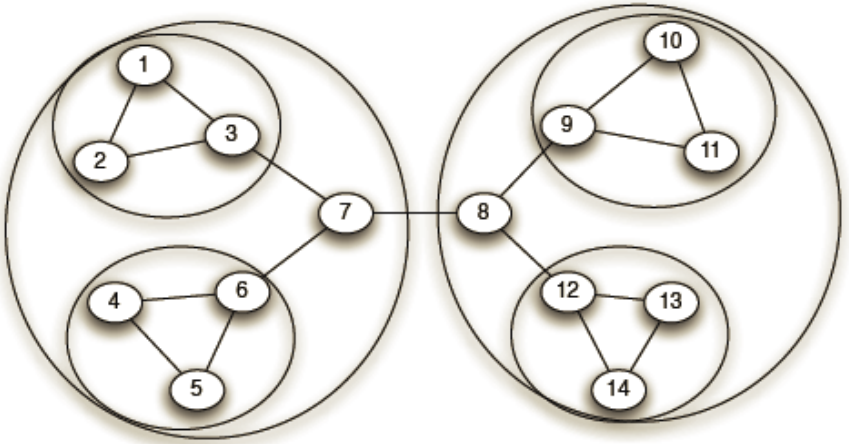
Step 2:



Step 3:



Hierarchical network decomposition:

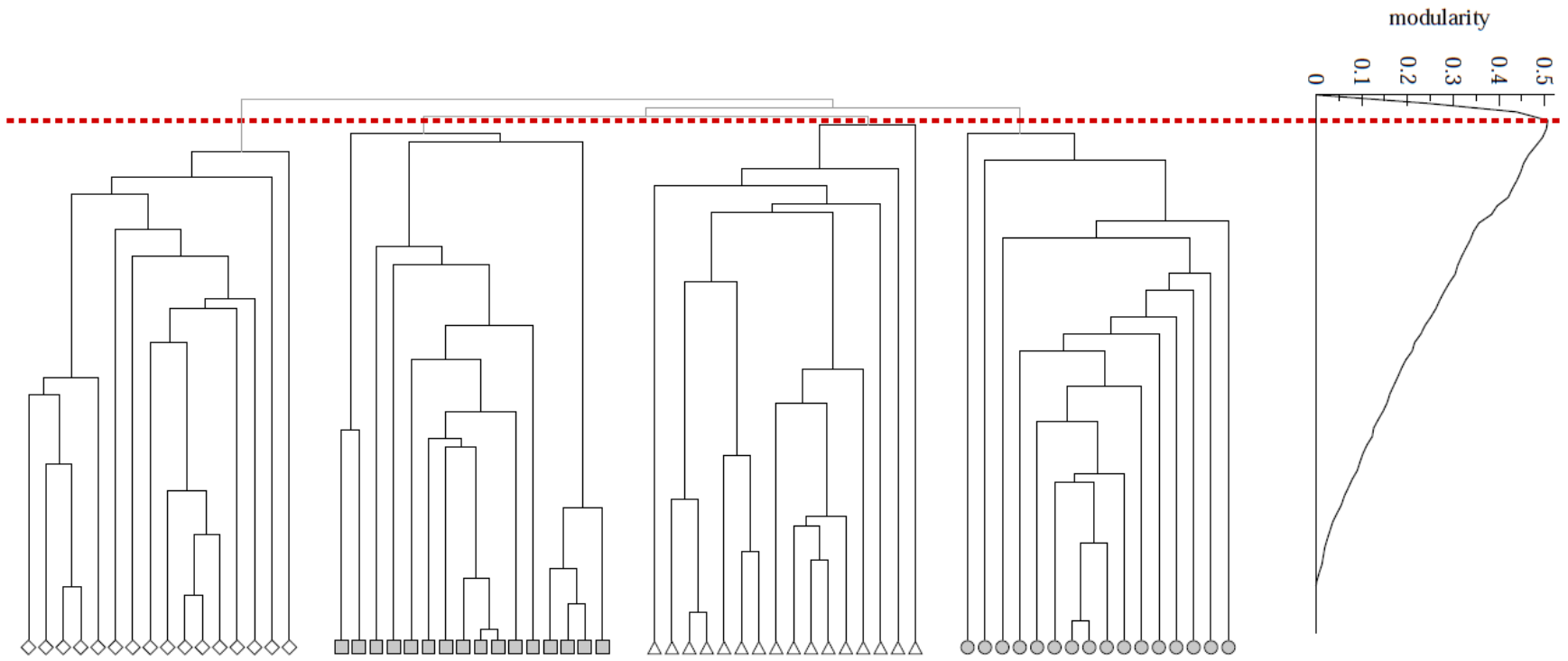


How to Select # of Clusters?

- Similar to agglomerative, with a different metric: **Modularity (Q; [-1, 1])**

- Idea: compare fraction of edges within a group to the fraction that would be observed for random connections

Selecting # of Clusters



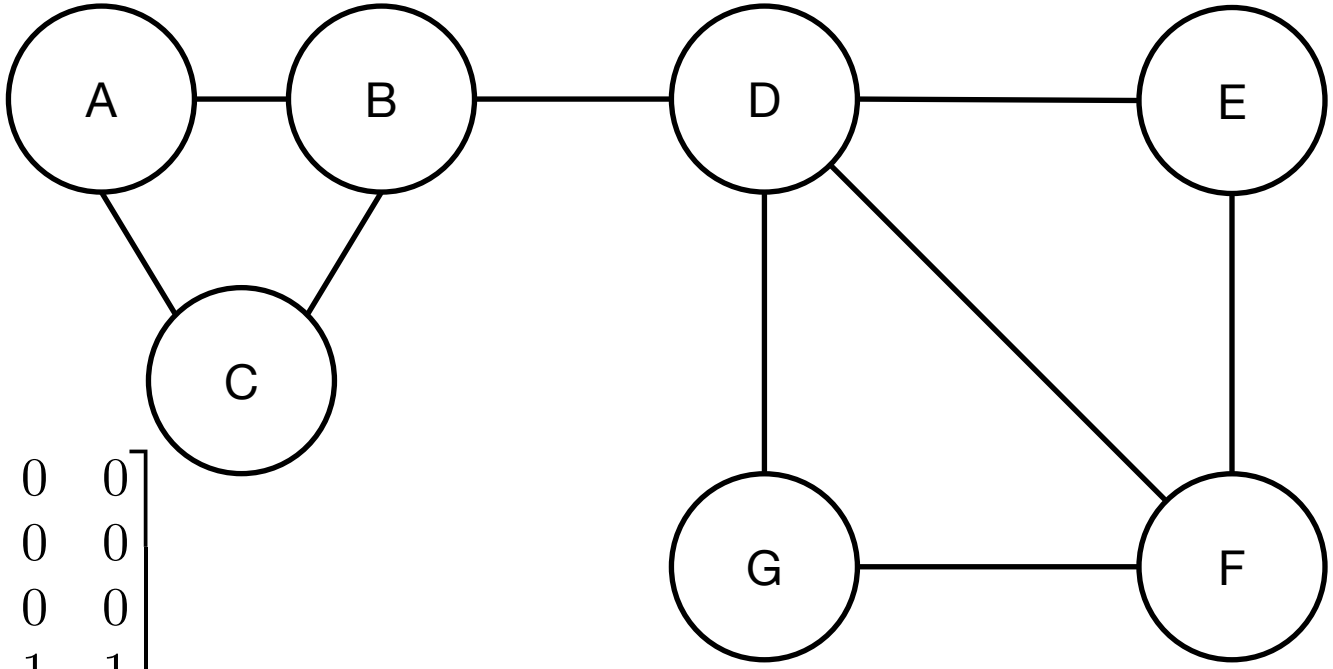
Direct Partitioning

- We now look at an approach to divide a graph into two disjoint groups, the task of **bi-partitioning**, via **spectral analysis**

- To do so, we must express the graph as a matrix

Adjacency Matrix (A)

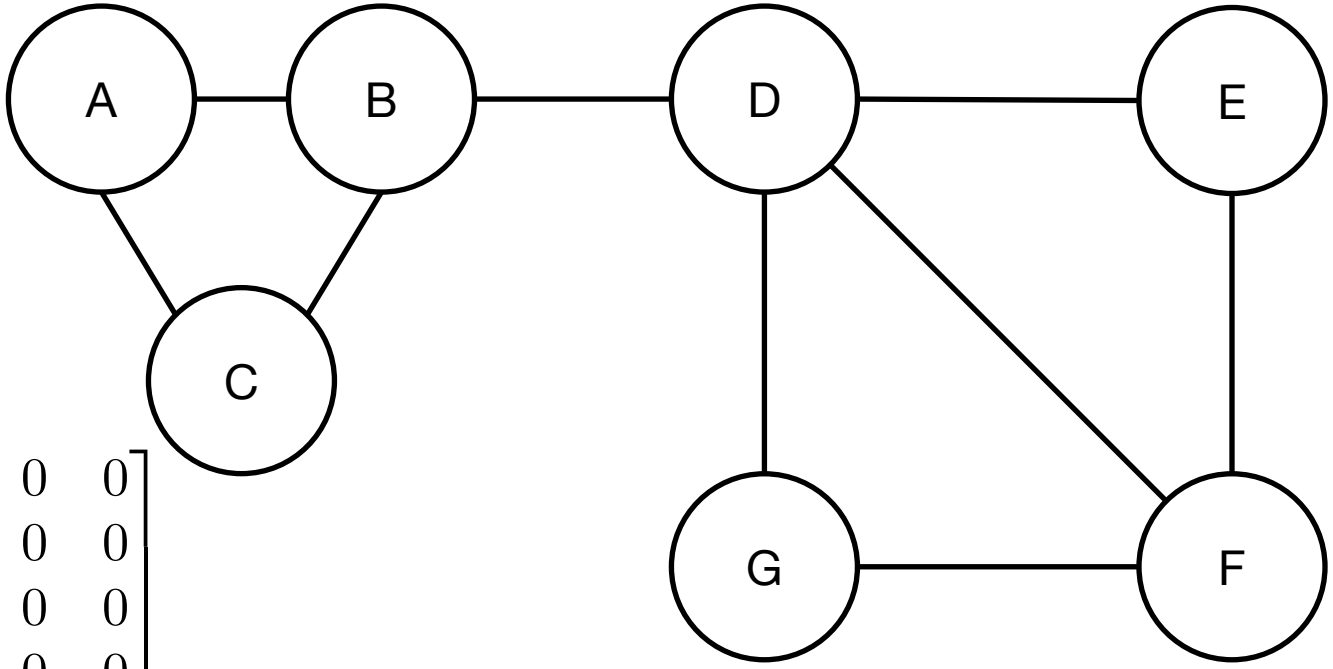
A_{ij} =1 if (i,j) is an edge, else 0



$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Degree Matrix (D)

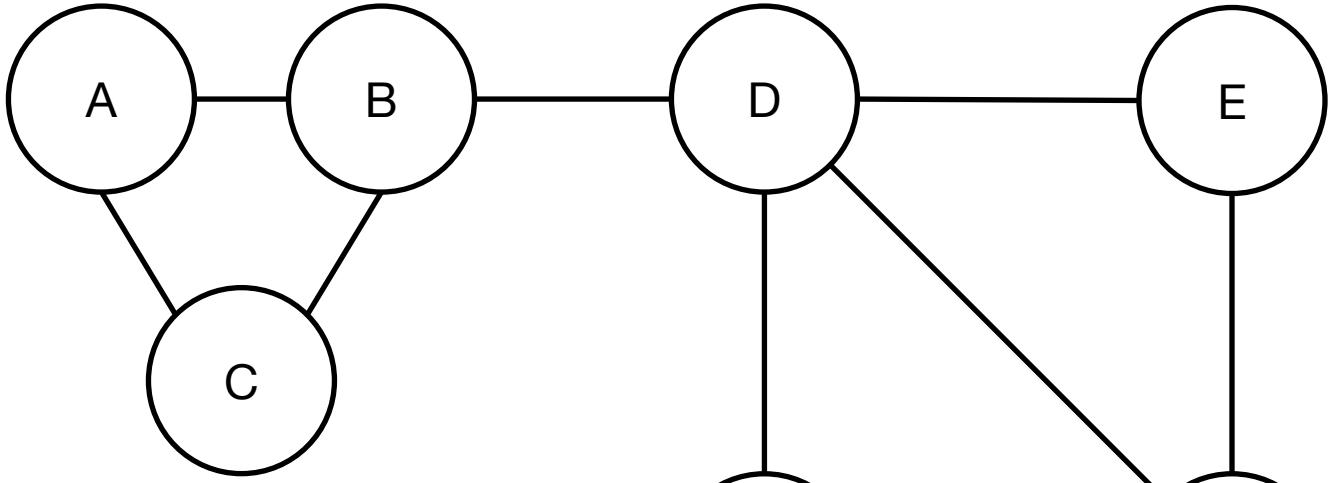
D_{ii} = deg(i), else 0



$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Laplacian (L)

$L = D - A$



$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 4 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

Checkup

- Rows sum to… ?
  0 (degree – edges)
- Columns sum to… ?
  0 (degree – edges)
- Symmetric?
  Yes! (D is diagonal, A is symmetric for **undirected** graph)
- Diagonal Dominant?
  Yes! (edge case = lonely)

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 4 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

The Plan

- We will now analyze the eigen-decomposition ($Lv = \lambda v$) of the Laplacian

- It turns out that the smallest eigen-values/vectors can tell us a lot!
  - Note: L is PSD, so $\lambda \geq 0$
  - 0: connectivity
  - Next smallest: partitioning

Checkup

- What is a non-trivial solution to the equation Lv = λv for λ=0 (i.e. Lv=0)?

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 4 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

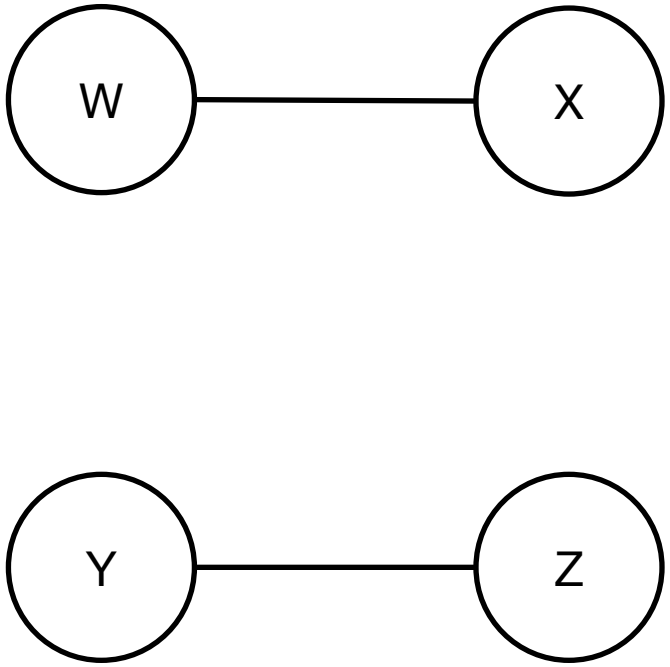
Answer

- What is a non-trivial solution to the equation Lv = λv for λ=0 (i.e. Lv=0)?
- You already solved it: 1's (column vector of constants)
- ACTUALLY, not the whole story…

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 4 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

Checkup

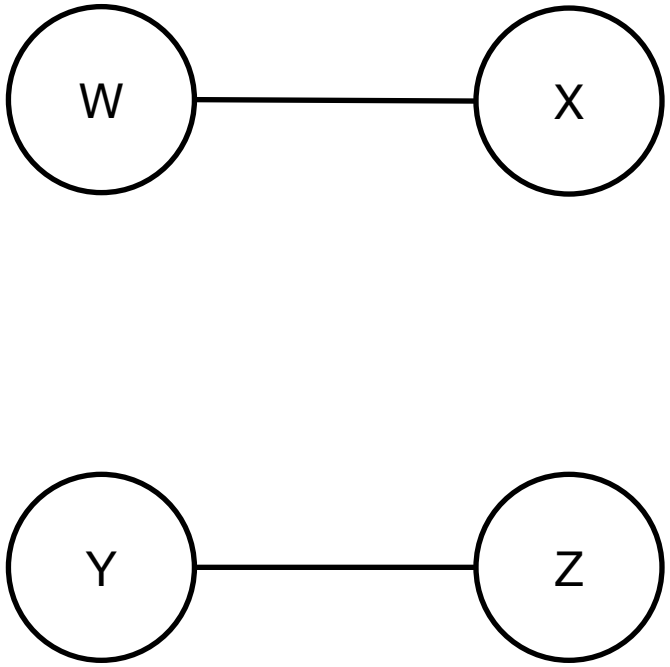
- Laplacian of the following graph?



Answer

- Laplacian of the following graph?

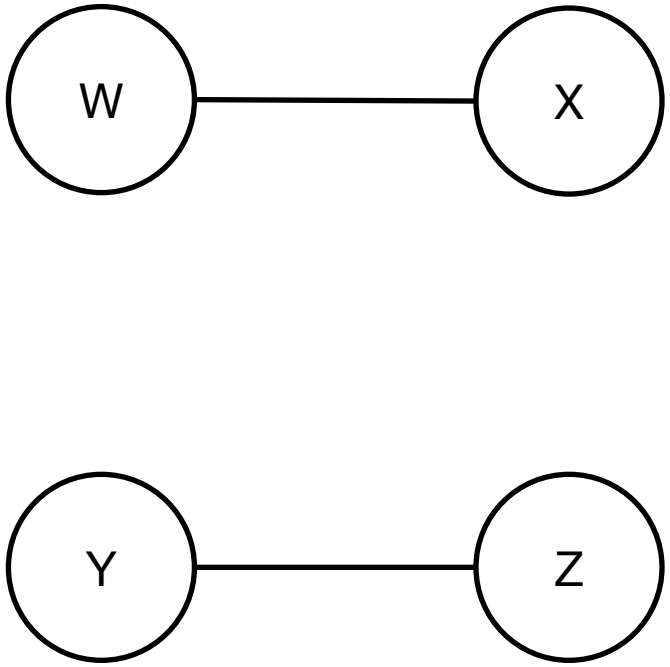
$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$



Checkup

- Null-space of L? (i.e. Lv=0)

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$



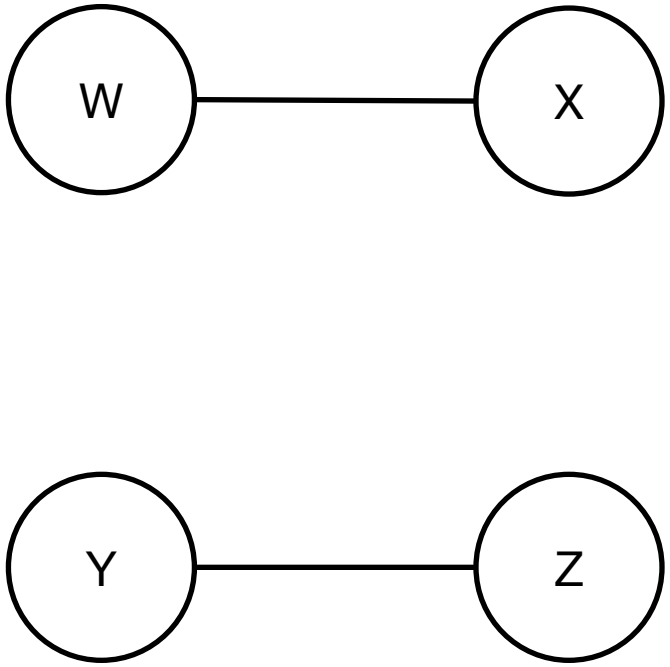
Answer

- Null-space of L? (i.e. Lv=0)

  $[1\ 1\ 0\ 0]^T$

  $[0\ 0\ 1\ 1]^T$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$



Connectivity via 0^{th} Eigenvalue

- If nodes i and j are connected, their values in the corresponding eigenvector must be equal
  - This "cancels out" across all rows
- So if the graph is connected, the 0^{th} eigenvector $[c\ c\ c\ \ldots\ ]^T$
  - Transitivity!
- Otherwise, null-space of L provides connected components
  - Could be detected by a simple clustering algorithm (e.g. k-Means)

Second-Smallest Eigenvalue

- Let's call this λ_1 (with corresponding v_1)
- If λ_1 is 0, what do we know?

  That there's not much sense in bi-partitioning :)
- Foreshadowing: from here, we'll construct an objective function for a bi-partitioning, and it will turn out to be minimized precisely by v_1

Setup

- For a given graph, we want to assign each node to one of two groups
  - Let's represent this assignment, per node, as the value -1 or +1 for variable n_i
  - So, a vector, n, of length |V| of either -1 or 1
- And do so in a way that minimizes the number of graph edges between the groups - so let's minimize…

$$(n_i - n_j)^2 \text{ for all edges } (i, j)$$

Checkup

What is $(n_i - n_j)^2$ for two connected nodes that …

- Are in the same group?
  $(1 - 1)^2 = 0$; $(-1 - -1)^2 = 0$
- Are in different groups?
  $(1 - -1)^2 = 4$; $(-1 - 1)^2 = 4$

So minimizing the sum across edges also minimizes cross-group edges – woohoo!

Converting to Matrices

- So we have $\sum(n_i - n_j)^2$
  - Or: $\sum (n_i^2 - 2n_in_j + n_j^2)$

- The way to express this in terms of the Laplacian is… $\mathbf{n^TLn}$
  - Each node will have $n_i^2 d(n_i)$ term (diagonal), so one n_i^2 per edge = $n_i^2 + n_j^2$
  - Each edge will have two $(-1)(n_in_j)$ terms (symmetric) = $-2n_in_j$
  - Each non-existent edge will multiply by zero from A_{ij} being zero

Checkup

- What is the easiest way to minimize the following function: $n_i^2 - 2n_in_j + n_j^2$

Answer

- What is the easiest way to minimize the following function: $n_i^2 - 2n_in_j + n_j^2$

- n = [0 0 … 0]
  - So we need to force values…

Forcing Non-Trivial Solutions

- Easy: $\sum n^2 = |V|$
  - The sum of the assignments must equal the total number of nodes

- Or, in vector notation $n^T n = |V|$

Objective Function – Minimize!

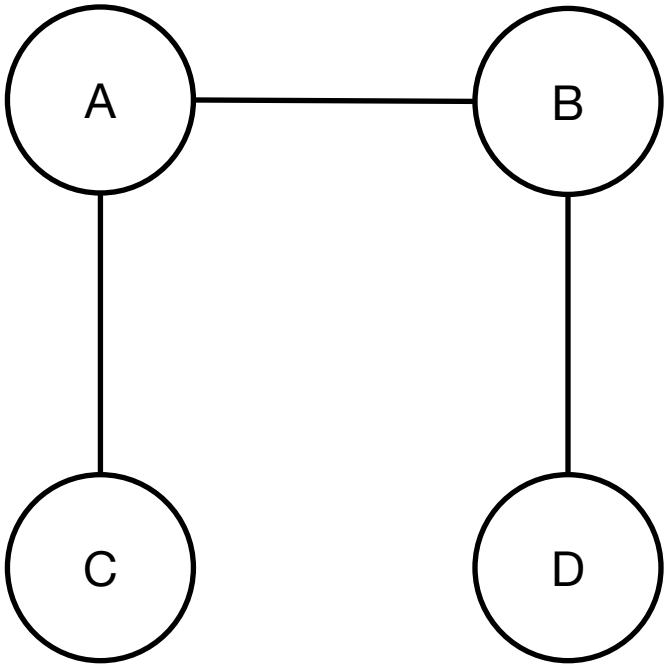
$$n^\intercal Ln + \lambda(n^\intercal n - |V|)$$

$$Ln - \lambda n = 0$$

SO, to find a good partitioning we need to...

- Find an eigenvector…
- with the second-smallest eigenvalue
  - Why doesn't 0 work?

Example (1): Graph



Example (1): Laplacian

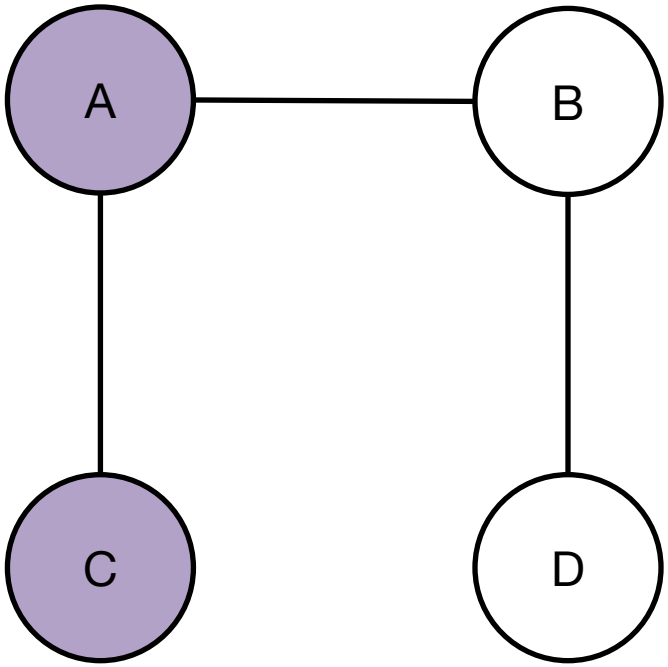
$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Example (1): Eigendecomposition

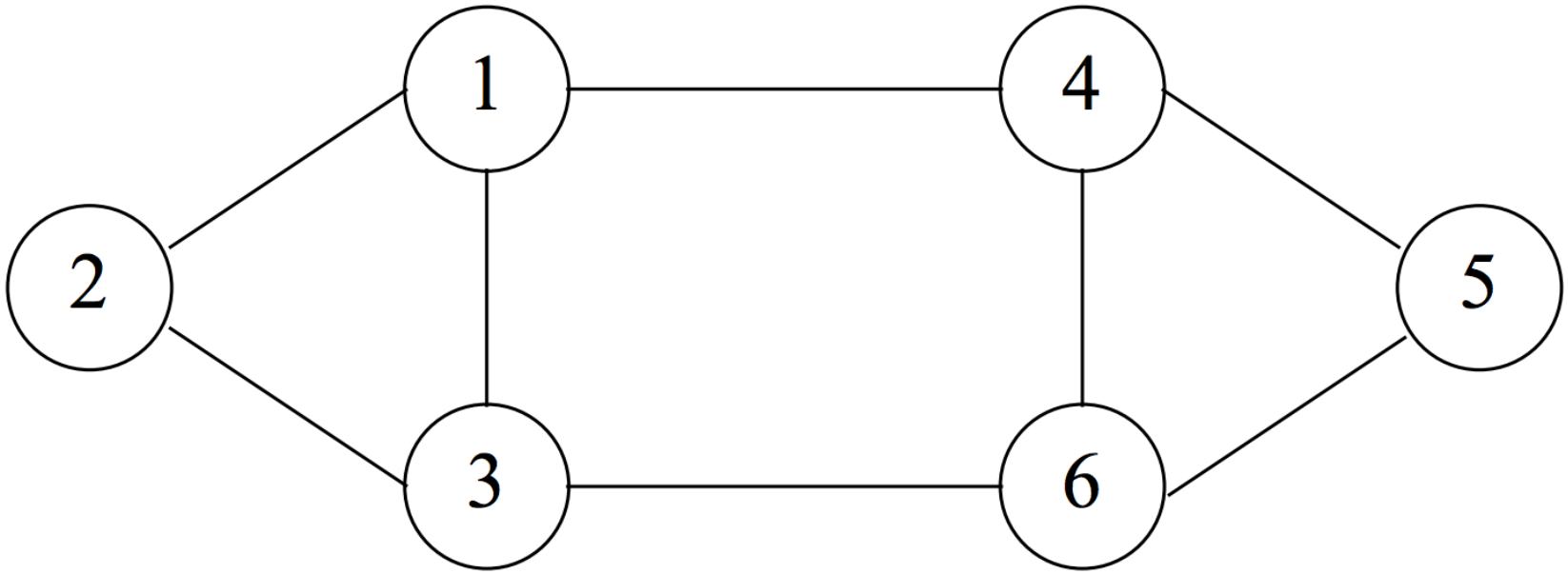
```
In [45]: np.linalg.eigh(A5)

Out[45]: (array([ -1.76298406e-16,   5.85786438e-01,   2.00000000e+00,
                   3.41421356e+00]),
          array([[-0.5       , -0.27059805,  0.5       , -0.65328148],
                 [-0.5       ,  0.27059805,  0.5       ,  0.65328148],
                 [-0.5       , -0.65328148, -0.5       ,  0.27059805],
                 [-0.5       ,  0.65328148, -0.5       , -0.27059805]]))
```

Sign = Partitioning (+ vs -)
+ {B, D}
- {A, C}

Example (1): Partitioning



Example (2): Graph



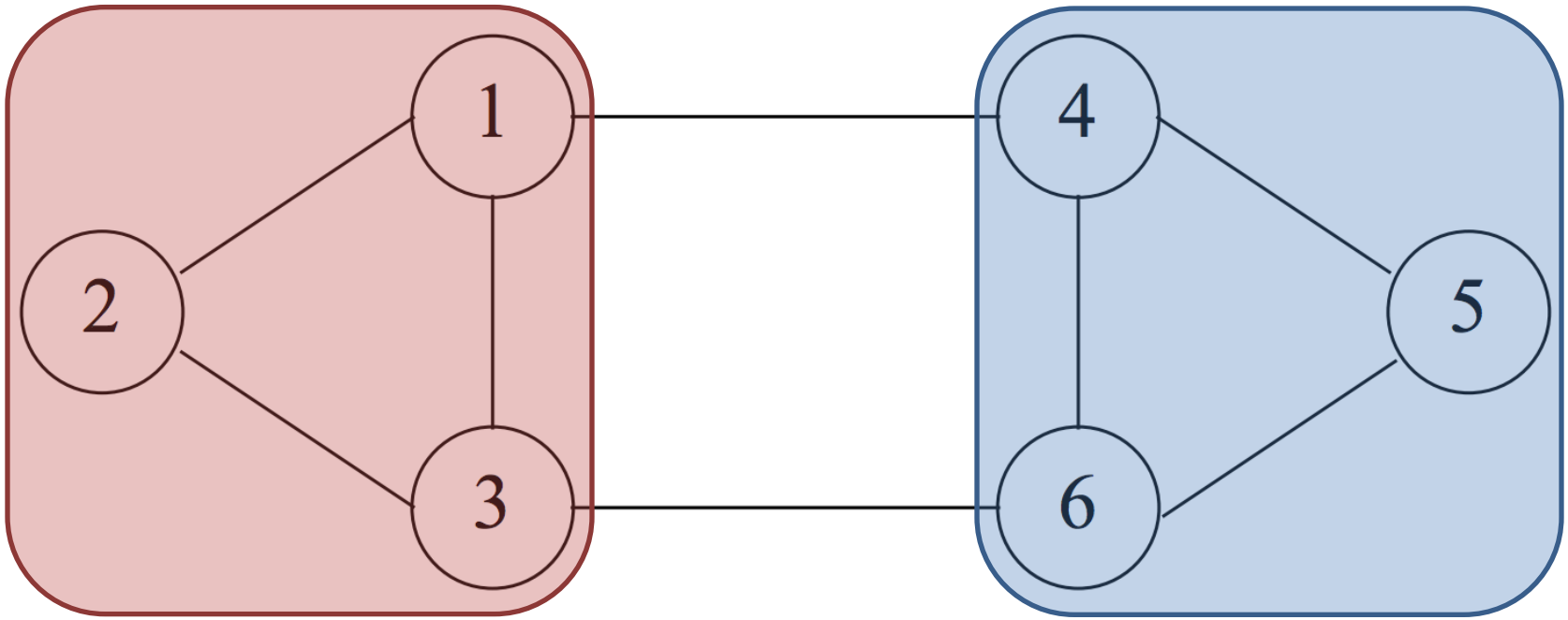
Example (2): Laplacian

$$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

Example (2): Eigendecomposition

Eigenvalue	0	1	3	3	4	5
Eigenvector	1	1	−5	−1	−1	−1
	1	2	4	−2	1	0
	1	1	1	3	−1	1
	1	−1	−5	−1	1	1
	1	−2	4	−2	−1	0
	1	−1	1	3	1	−1

Example (2): Partition



Many Other Topics

- Overlapping communities
- Network/Node properties
  - Triangles, neighborhoods
  - Centrality, influence
- Network formation
- Problems
  - Link prediction