How are we doing today?

We'll start w/ reviewing your Lec 11 ICAs !! :)

Admin

At 10am: your Test 1 was 67% graded → grades to you all by Monday

- For my ICAs for my lectures, we are moving to the following format:

- Every lecture, you will answer *the same* three questions:

1. What did you learn from this lecture?

2. What are you confused about?

3. ~~(a question about either an ICA or a homework problem)~~

- I will stop lecture 10 minutes early for you to do this. You are expected to do this during class time.

- **Only** turn this in on Canvas

Admin

- You may assume that there will be no lecture content on test days
- You may assume that there will be no asynchronous lecture content for Patriot's Day (this is the last Monday holiday this semester) -> we will have 1 fewer ICA than Prof. Higger's sections
  - Because you should all be watching Molly Seidel in the marathon
- Plan to be here in person for the following days before spring break:
  - Thursday, March 3rd (Test 2)
  - Thursday, March 10th (Mini Project Day 1)

line of best fit, eigenvalues/vectors, Markov Chains review

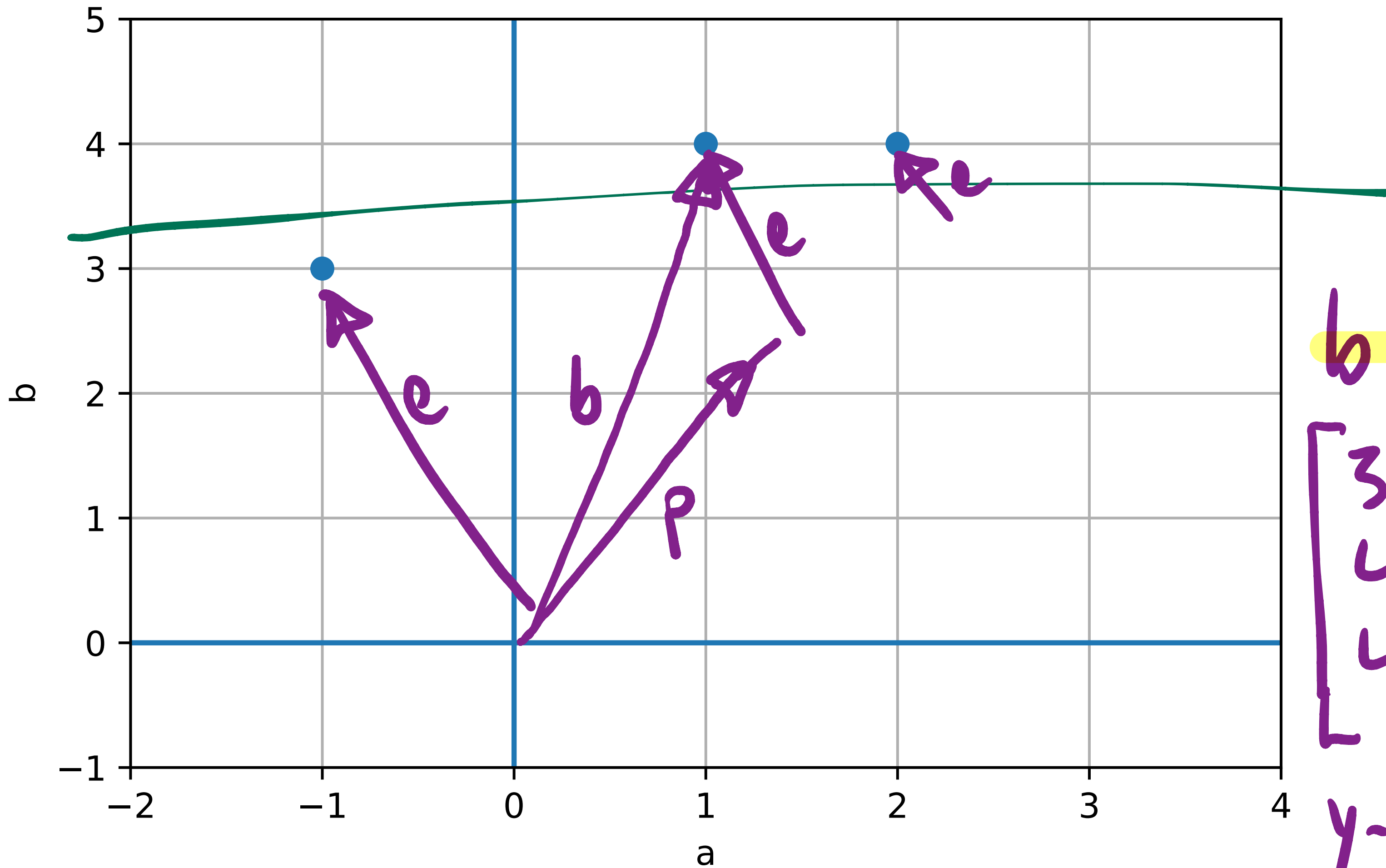
↳ HW 4

Line of Best Fit - Lec 11 ICA

$\vec{p} = \frac{b^T a}{a^T a} a$

- Find the line of best fit for the below scatter points. (Using $p = ma$)

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



$b = ma$

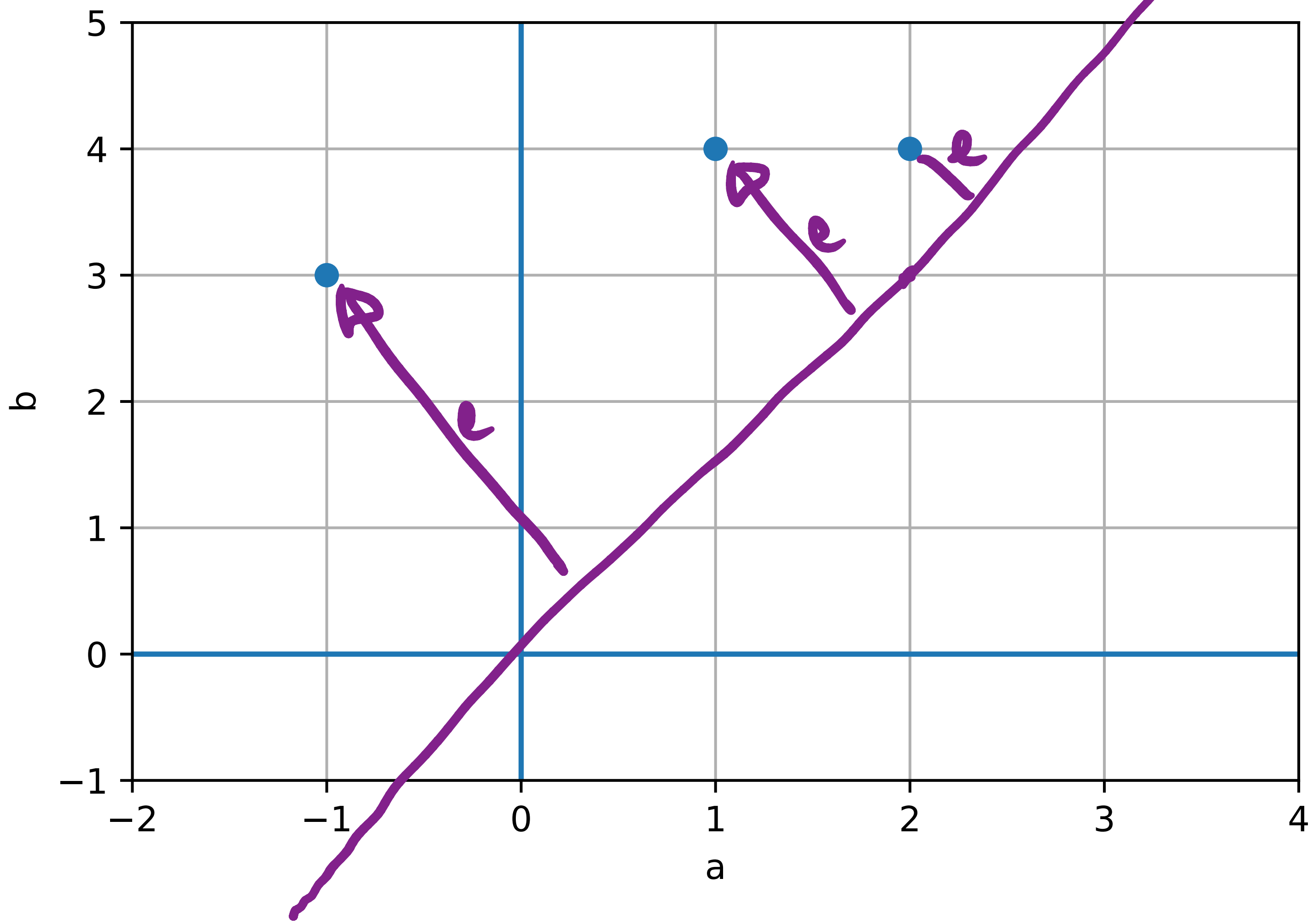
$$\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} = m \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

y-values x-values

$y = mx$

Line of Best Fit - Lec 11 ICA

- Find the line of best fit for the below scatter points. (Using $p = ma$)



$$\begin{bmatrix}-1\\3\end{bmatrix}, \begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}$$

$$\vec{p} = \frac{\begin{bmatrix}3\\4\\4\end{bmatrix}\begin{bmatrix}-1 & 1 & 2\end{bmatrix}}{\begin{bmatrix}-1\\1\\2\end{bmatrix}\begin{bmatrix}-1 & 1 & 2\end{bmatrix}} \begin{bmatrix}-1\\1\\2\end{bmatrix}$$

$$= \frac{a}{6}\begin{bmatrix}-1\\1\\2\end{bmatrix} = \frac{3}{2}a$$

Line of Best Fit - Equation Summaries

vector

- To find a polynomial of best fit, you'll be solving for: $\overrightarrow{p} = A\vec{m}$, using the equation $\overrightarrow{p} = A(A^TA)^{-1}A^Tb$ (since you know the values of b and a)

- Where b is the vector of "y values"; A is the matrix of "x values" raised to the power of the "current" coefficient; m is the vector of "slopes"

$$\underbrace{\underbrace{\begin{bmatrix}3\\4\\4\end{bmatrix} = \begin{bmatrix}-1^0\\1^0\\2^0\end{bmatrix} m_0 + \begin{bmatrix}-1^1\\1^1\\2^1\end{bmatrix} m_1}_{b = m_0 a^0 + m_1 a^1} + \begin{bmatrix}-1^2\\1^2\\2^2\end{bmatrix} m_2}_{b = m_0 a^0 + m_1 a^1 + m_2 a^2} + \dots$$

Eigenvalues/vectors - Lec 11 ICA

- What is an eigenvector?

↳ a vector that for a given transformation doesn't move off its own span

- What is an eigenvalue?

↳ how much eigenvector scales

- Find the eigenvalues/vectors for the given matrix: $A = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}$

$$A\vec{v} = \lambda\vec{v}$$

eigenvectors

eigenvalue

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} v \end{bmatrix}$$

Eigenvalues/vectors - Lec 11 ICA

- Find the eigenvalues for the given matrix:

$$A = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}$$

$A\vec{v} = \lambda\vec{v}$

$A\vec{v} - \lambda\vec{v} = 0$

$\underbrace{(A - \lambda I)}_{\text{to be } 0}\vec{v} = 0 \rightarrow \det(A - \lambda I) = 0$

$\lambda = -1, 2$

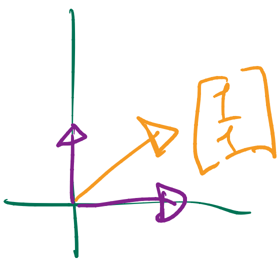
$$\det\left(\begin{bmatrix} -1-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

$(-1-\lambda)(2-\lambda) - (3)(0) = 0$

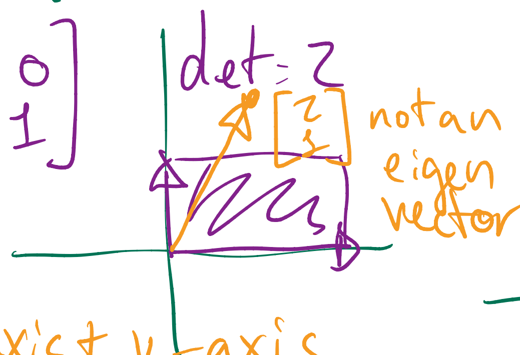
$(-1-\lambda)(2-\lambda) = 0$

What's going on w/ Eigen vectors + determinants of zero? (added after lecture)

- the only way for a matrix * a non-zero vector to be 0 is for the determinant of the matrix to be 0
  - ↳ this means that the transformation w/ this matrix "squishes into lower dimension space.

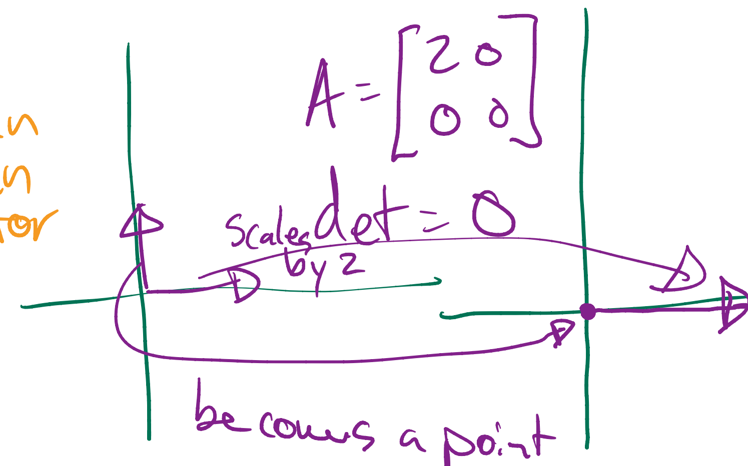


$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$



eigen vectors: x axis + y-axis

$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$



$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\det(A - I\lambda) = 0$

- notice that we have an eigen value associated w/ each basis vector

$\Rightarrow \lambda = 2, 1$

- when we ask for $\det(A - I2)$, we are essentially "reducing out" the basis vector associated w/ this eigen value:

$$\begin{bmatrix} 2-2 & 0 \\ 0 & 1-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ tells us the x-axis is an eigenvector

Eigenvalues/vectors - Lec 11 ICA

- Find the eigenvectors for the given matrix:

$$A = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}$$

$\lambda = -1, 2$

$(A - I\lambda)\vec{v} = 0$

$$\underbrace{\begin{bmatrix} -1 - -1 & 3 \\ 0 & 2 - -1 \end{bmatrix}}_{A - I\lambda} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} 3 \\ 3 \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\boxed{\text{eigenvector(s): anything on the x axis}}$

Eigenvalues/vectors - Lec 11 ICA

Added after lecture

- Find the eigenvectors for the given matrix:

$$A = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} -1-2 & 3 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x + 3y = 0 \Rightarrow -x + y = 0 \Rightarrow x = y$$

Eigenvectors:

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ + all scaled versions

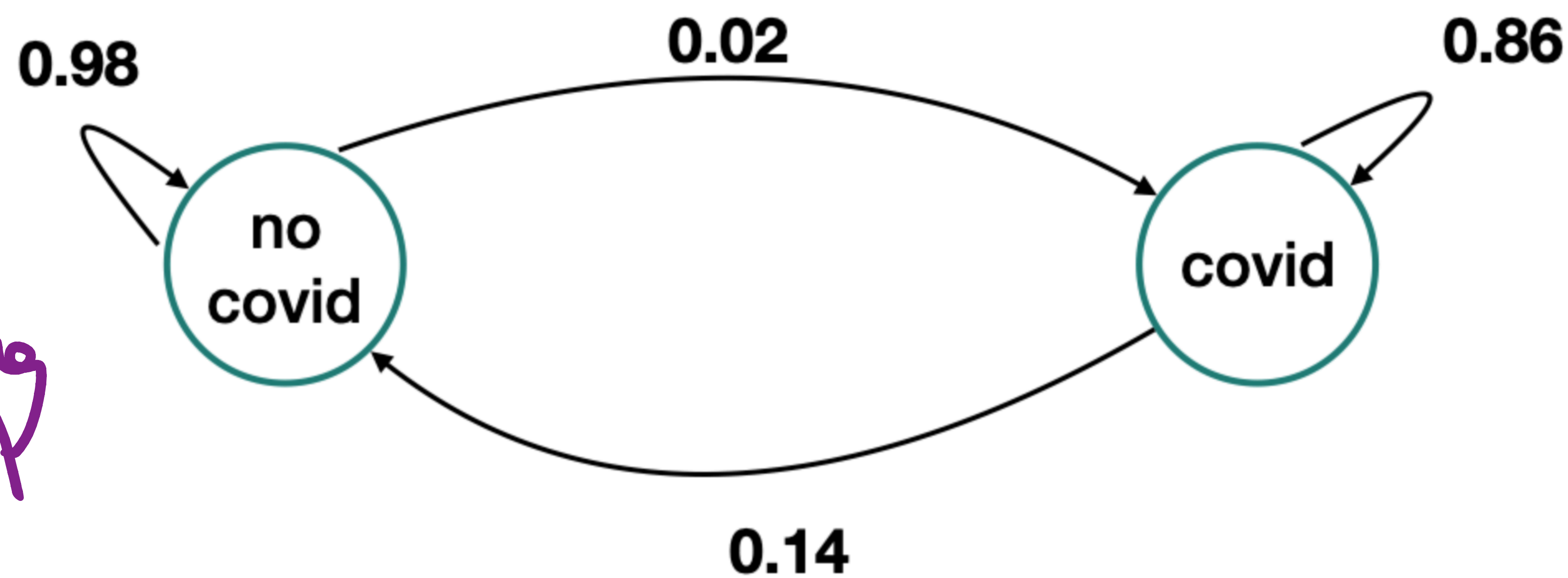
Eigenvalues/vectors - Lec 11 ICA

- For HW 4—you'll need to find the determinant of a 3 x 3 matrix. This is the equation:
- $$det(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}) = a * det(\begin{bmatrix} e & f \\ h & i \end{bmatrix}) - b * det(\begin{bmatrix} d & f \\ g & i \end{bmatrix}) + c * det(\begin{bmatrix} d & e \\ g & h \end{bmatrix})$$

- (Feel free to look up examples online/in your textbooks too :D)

Markov Chains - Lec 11 ICA

what is time?
↳ according to model def.



$$\begin{bmatrix} c' \\ n' \end{bmatrix} = \begin{bmatrix} 0.86 \\ 0.14 \end{bmatrix} c + \begin{bmatrix} 0.02 \\ 0.98 \end{bmatrix} n$$

today state

$$x' = \begin{bmatrix} 0.86 & 0.02 \\ 0.14 & 0.98 \end{bmatrix} x$$

overall state for tomorrow

Markov Chains - Lec 11 ICA

- At time 3:

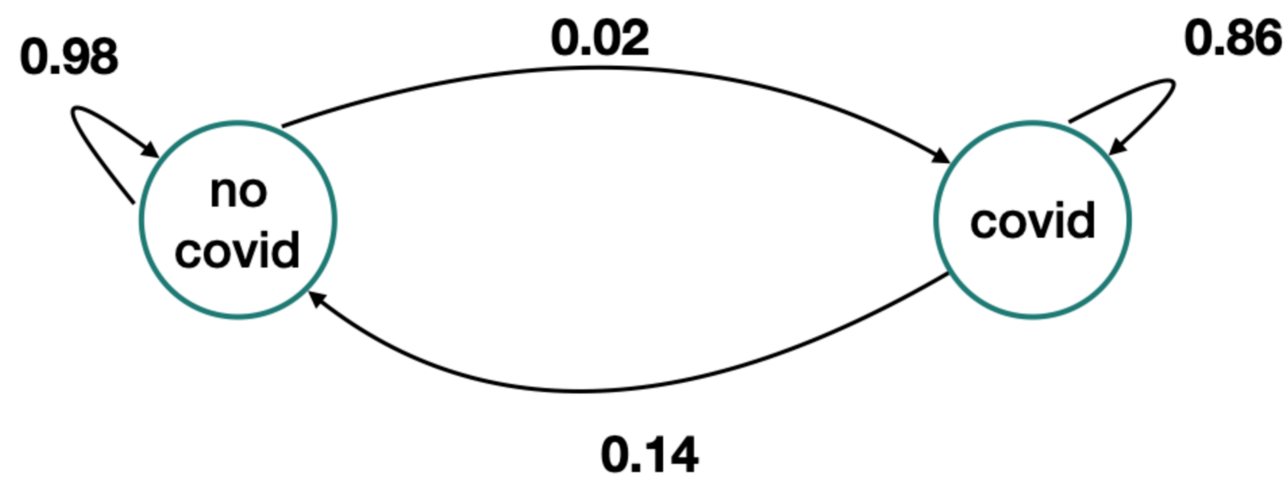
```
x = np.array([[50, 10000]]).T
x_prime_3 = A @ A @ A @ x
print(x_prime_3)
```

```
[[ 541.3008]
 [9508.6992]]
```

$AAAx = x^3$

- At time 4:

```
[[ 655.692672]
 [9394.307328]]
```



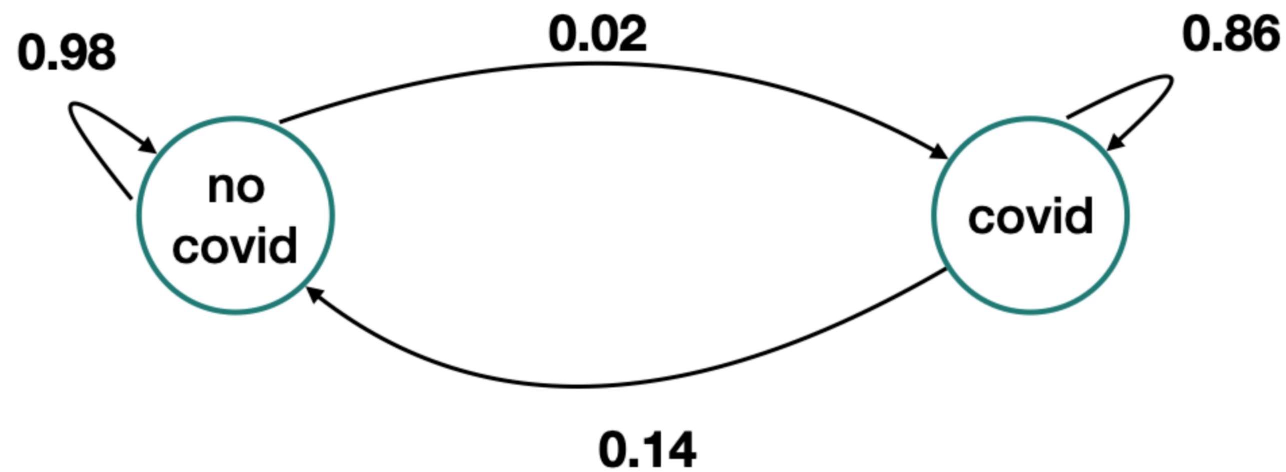
- are covid cases increasing? → yes!

Markov Chains - Lec 11 ICA

- For these systems, there is a notion of a "steady state". If we find the vector associated with eigenvalue 1, this translates to the "steady state"

```
values, vectors = np.linalg.eig(A)
print(values)
print(vectors[:, 0]) # first eigen vector
print(vectors[:, 1]) # second eigen vector
```

```
[0.84 1.  ]
[-0.70710678  0.70710678]
[-0.14142136 -0.98994949]
```




```
# steady state
print (A @ vectors[:, 1])
print (A @ A @ vectors[:, 1])
print (A @ A @ A @ vectors[:, 1])
```

```
[-0.14142136 -0.98994949]
[-0.14142136 -0.98994949]
[-0.14142136 -0.98994949]
```

Markov Chains - Lec 11 ICA

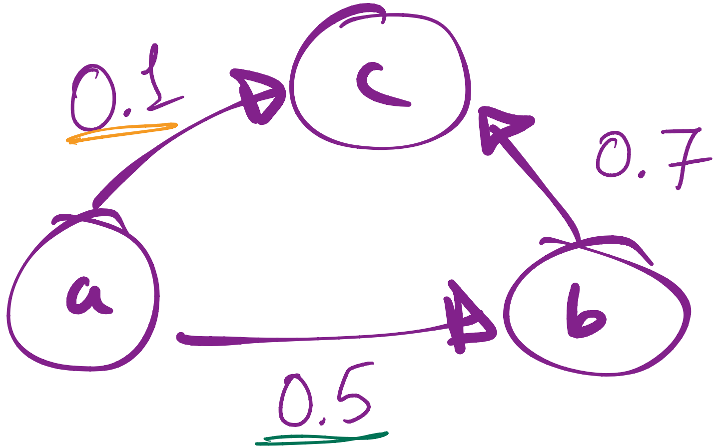
- How can we make these numbers make more sense? (We don't have negative fractional people, last I checked)

```python
values, vectors = np.linalg.eig(A)
print(values)
print(vectors[:, 0]) # first eigen vector
print(vectors[:, 1]) # second eigen vector
```

```
[0.84 1.  ]
[-0.70710678  0.70710678]
[-0.14142136 -0.98994949]
```

$$\begin{array}{c} c \\ n \end{array} \begin{bmatrix} 0.7 \\ -0.99 \end{bmatrix} = \begin{bmatrix} \dfrac{0.7}{(0.7+-0.99)} \\ \dfrac{-0.99}{(0.7+0.99)} \end{bmatrix} = \begin{bmatrix} -2.4 \\ 3.4 \end{bmatrix}$$

$$\begin{array}{c} c \\ n \end{array} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$



12:36

break: 12:40

↳ stretch

↳ compare w/ neighbors

↳ re-set yourselves a bit

$$a' = a - \underline{0.5a} - \underline{0.1a}$$

$$b' = b - 0.7b + 0.5a$$

$$c' = c + 0.1a + 0.7b$$

Felix reminders:

- ↳ add 2^{nd} eigenvector math
- ↳ explicit example of why det(~~~) = 0 makes eigenvectors happen

intro to probability and statistics

HW5 boundary

What is the probability of rolling two ones on two 6-sided dice?

AND $\frac{1}{6} * \frac{1}{6} = \frac{1}{36}$

What is the probability of rolling a prime number on one 6-sided die?

$\rightarrow$ 2, 3, 5

OR $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$

Random Variables

- A **random variable** is a variable whose value is determined by a probability distribution.

- Examples!

↳ a die $\{1/6, \ldots 1/6\}$

↳ the weather tomorrow : $\{$snow: $1/2$, sleet: $1/4$, fog: $1/4\}$

↳ deck of cards

Random Variables

- Random variables are normally written with capital letters: X, Y, Z

X = outcome of rolling a 6-sided die

Expected Value

- A **random variable** is a variable whose value is determined by a probability distribution.

- Random variables have an **expected value**. (written as: $E[X]$)

- When I roll a 6-sided die, what value do I expect to see?

5, 3, 2, 1, 3 . . . . .

Expected Value

- A **random variable** is a variable whose value is determined by a probability distribution.

- Random variables have an **expected value.** This corresponds to the average value I expect given **infinite trials**. (written as $E[X]$)

- When I roll a 6-sided die from **now until eternity**, what **average** value do I expect to see?

$$\frac{1}{6}(1) + \frac{1}{6}(2) + \frac{1}{6}(3) + \frac{1}{6}(4) + \frac{1}{6}(5) + \frac{1}{6}(6)$$

$$= \frac{1 + 2 + \ldots + 6}{6} = 3.5$$

Expected Value

- The expected value equation is, formally: $E[X] = \sum_{x} P(X = x) * x$

little x is a specific outcome

Expected Value - ICA Question 1

If a random variable, X, is a 5-sided die, what is the **expected value**? ($E[X]$)

$$E[X] = 3$$

... and if a trickster erased all odd numbers from the die? (making these sides blank instead)

$1, 3, 5 \rightarrow 0$

$$\frac{0+2+0+4+0}{5} = \frac{6}{5}$$

Variance

- A **random variable** is a variable whose value is determined by a probability distribution.
- Random variables have a **variance**. This is a measurement of the average distance from the **expected value** each individual trial will be.
- When I roll a 6-sided die, <u>**how far**</u> from the **expected value** do I think that it will be?

$$\frac{1}{6}(1-3.5)^2 + \frac{1}{6}(2-3.5)^2 + \dots\dots + \frac{1}{6}(6-3.5)^2$$

"distance" 2.92

Variance: 2.92

Variance

• std dev is $\sqrt{Var(X)}$

- The variance equation is, formally: $E[(x - E[X])^2]$
- Alternatively, we can write this as: $Var(X) = \sum_x P(X = x) * (x - E[X])^2$
- Felix comes to you with a coin that has their favorite numbers on each side: 97 and -40. What is the variance?

$$E[X] = \frac{1}{2}(97) + \frac{1}{2}(-40) = \frac{57}{2} = 28.5$$

$$Var(X) = \frac{1}{2}(97-28.5)^2 + \frac{1}{2}(-40-28.5)^2 = \text{the answer}$$

Expected Value - ICA Question 2

If a random variable, X, is a 5-sided die, what is the **variance**? ($Var(X)$)

$Var(X) = 2$

What can you do to the die to reduce the variance but maintain $E[X]$?

↳ make all sides = 3, $E[X] = 3$ $Var(X) = 0$

↳ $5 \to 4$, $1 \to 2$

Linearity of Expectation

- **Linearity of expectation** says that given multiple random variables, we can sum their expected values to get an overall expected value.
- Say that we want to know the expected value of rolling 3 dice.
- Each die has 3.5, so $E[X+X+X] =$ $E[X] + E[X] + E[X]$

$= 10.7$

$E[X]$

$X = 6$-sides

$Y = 5$-sides

$E[X+Y] = 3.5 + 3 = 6.5$

Linearity of Expectation

- **Linearity of expectation** says that given multiple random variables, we can sum their expected values to get an overall expected value.
- Linearity of expectation also applies to variance.
- Say that we want to know the variance of rolling 3 6-sided dice:
- Each die has $Var(X) = 2.92$, so $Var(X + X + X) =$ 8.76

$$Var(X+Y) = 2.92 + \underset{\uparrow}{\underset{Var(Y)}{2}} = 4.92$$

Independence Alert!

- We say that two random variables are **independent** if they have declared autonomy under the charter of Turtles Great and Small.

- If variables are independent, everything that we've said is **true**.

- If variables are dependent, linearity of expectation/variance **does not hold**.

Independence Alert!

- Wait, how do we know if random variables are independent?
- Their outcomes don't depend on each other.
  - Random variables X and Y are independent if when X's value changes, that doesn't affect the probability of getting a particular outcome y for Y. (And vice-versa)

Independence Alert!

- Independent things:

↳ 2 dice → a coin + a die

↳ 2 decks of cards

- Dependent things:

↳ 2 draws w/o replacement from a deck

↳ the weather today + amount of HW I do

Independence - ICA Question 3

random

With your neighbor(s), come up with 2 new examples of variables that are independent and two new examples of variables that are dependent.

random

ind	dep
- coin flips	- covid + absences
- height + eye color	- weather today + tomorrow

Schedule

Turn in ICA 12 on **Canvas (not on Gradescope)** → access code: "tree"

HW 4 is due on Sunday

TEST 2 is in class the Thursday one week from today

Send me an email if you're feeling overwhelmed! (I know that there's a lot of work in this class, we will work with you to make sure that you don't fall behind) (I'm serious, please do this)

Mon	Tue	Wed	Thu	Fri	Sat	Sun
February 21st President's day! Asynchronous lecture to be done before class Thursday, Eigenvectors, dynamical systems/Markov	**Felix OH Calendly**	**Felix OH Calendly**	Lecture 12 - intro prob. and stats **Felix OH Calendly**			**HW 4 due @ 11:59pm**
February 28th Lecture 13 - law of large numbers, distributions **HW 5 out** →	**Felix OH Calendly**		**TEST 2 IN CLASS** ↳ same deal as Test 1			**HW 5 due @ 11:59pm**

More recommended resources on these topics

- YouTube, 3Blue1Brown: The determinant | Chapter 6, Essence of Linear Algebra
- YouTube, 3Blue1Brown: Eigenvectors and eigenvalues | Chapter 14, Essence of Linear Algebra
- Youtube: Khan Academy: Mean (expected value) of a discrete random variable | AP Statistics | Khan Academy
- Youtube: Khan Academy: Variance and standard deviation of a discrete random variable | AP Statistics | Khan Academy