CS2810 Feb 22

Day 11

(Linear) Dynamical System → FROM LAST CLASS
- ecology

Determinants

Eigenvalues & Eigenvectors
- Definition
- Finding eigenvalues
- Finding eigenvectors
- A matrix with distinct eigenvalues has linearly independent eigenvectors

Change of Basis and Eigenvectors

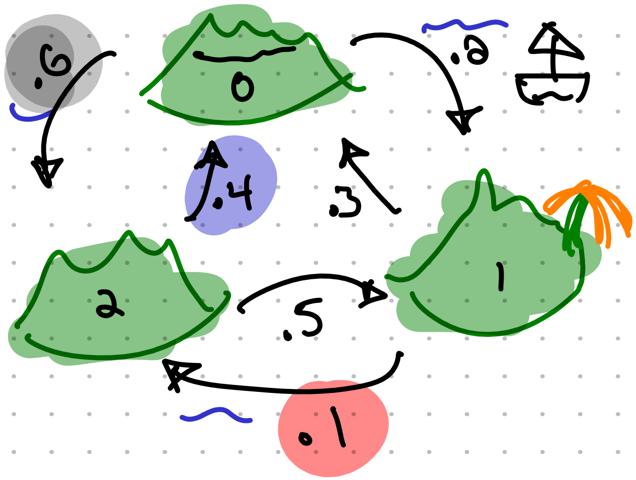
Math "Magic" Trick

People island 0 after 1 move

$$x_0' = x_0 - .6x_0 + .4x_2$$

$$x_1' = x_1 - .1x_1$$

$$x_2' = x_2 + .6x_0 - .4x_2 + .1x_1$$



$$\begin{bmatrix} x_0' \\ x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} .2 \\ .2 \\ .6 \end{bmatrix} x_0 + \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix} x_1 + \begin{bmatrix} .4 \\ .5 \\ .1 \end{bmatrix} x_2$$

$$x_0' = .2x_0 + .3x_1 + .4x_2$$

$$x_1' = .2x_0 + .6x_1 + .5x_2$$

$$x_2' = .6x_0 + .1x_1 + .1x_2$$

x' POP AFTER MOVE

A

POPULATION ON EACH ISLAND INIT X

$$\begin{bmatrix} x_0' \\ x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} .2 \\ .2 \\ .6 \end{bmatrix} x_0 + \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix} x_1 + \begin{bmatrix} .4 \\ .5 \\ .1 \end{bmatrix} x_2 = \begin{bmatrix} .2 & .3 & .4 \\ .2 & .6 & .5 \\ .6 & .1 & .1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

$$x' = Ax$$

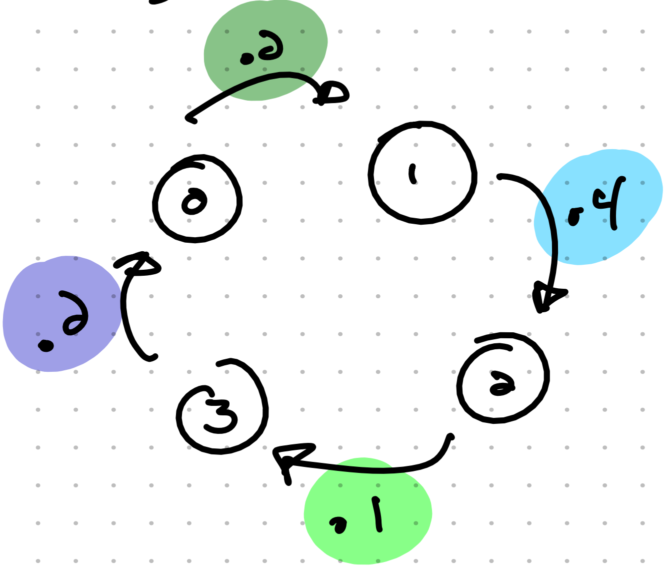
Dynamical System Question

- Why/When does it have steady state distribution?
- How can I find steady state distribution?

(More later)

ICA 1

WRITE MATRIX A WHICH REPRESENTS ISLAND POPULATION AFTER 1 STEP



$$x_0' = x_0 - .2x_0 + .2x_3$$

$$x_1' = x_1 + .2x_0 - .4x_1$$

$$x_2' = x_2 + .4x_1 - .1x_2$$

$$x_3' = x_3 + .1x_2 - .2x_3$$

$$x_0' = x_0 - .2x_0 + .2x_3$$
$$x_1' = x_1 + .2x_0 - .4x_1$$
$$x_2' = x_2 + .4x_1 - .1x_2$$
$$x_3' = x_3 + .1x_2 - .2x_3$$

$$x_0' = .8x_0 + .2x_3$$
$$x_1' = .2x_0 + .6x_1$$
$$x_2' = .4x_1 + .9x_2$$
$$x_3' = .1x_2 + .8x_3$$

$$\begin{bmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \\ 0 \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} 0 \\ .6 \\ .4 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0 \\ .9 \\ .1 \end{bmatrix} x_2 + \begin{bmatrix} .2 \\ 0 \\ 0 \\ .8 \end{bmatrix} x_3$$

$$x' = \begin{bmatrix} .8 \\ .2 \\ 0 \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} 0 \\ .6 \\ .4 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0 \\ .9 \\ .1 \end{bmatrix} x_2 + \begin{bmatrix} .2 \\ 0 \\ 0 \\ .8 \end{bmatrix} x_3 = \underset{A}{\begin{bmatrix} .8 & 0 & 0 & .2 \\ .2 & .6 & 0 & 0 \\ 0 & .4 & .9 & 0 \\ 0 & 0 & .1 & .8 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

Determinant 2×2 CASE

A value associated with each square matrix:
- A matrix with det = 0 has linearly dependent columns
  (some non-zero linear combination of columns = zero vector)

$\text{DET}(A) = 0 \leftrightarrow$ COLS OF A ARE DEPENDENT $\leftrightarrow \exists$ LIN COMBO OF COL OF A EQUAL TO 0

NOTATION

$$\text{DET}(A) = |A| = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc$$

2×2 CASE

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot -2 + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

DEPENDENT

Determinant (3x3 case)

$$|A| = \left|\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right| = aei + bfg + cdh - afh - bdi - ceg$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

EIGENVALUES + EIGENVECTORS

An eigenvector is a non-zero vector, v, associated with a square matrix
- multiplying Av scales v, but doesn't change its direction

OUTPUT IS SCALED VERSION OF V

$$A\vec{v} = \lambda\vec{v}$$

A MULTIPLIES V

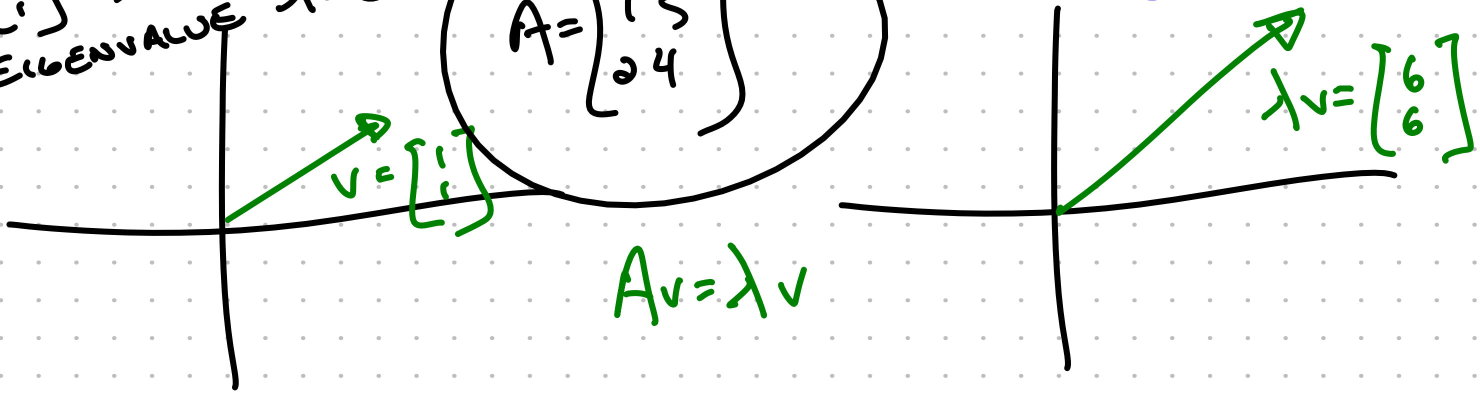
SCALE OF EIGENVECTOR V IS ITS ASSOCIATED EIGENVALUE

Geometry of Eigenvector | Eigenvalue

$v = \begin{bmatrix}1\\1\end{bmatrix}$ is eig vec w eigenvalue $\lambda = 6$

$A = \begin{bmatrix}1 & 5\\2 & 4\end{bmatrix}$

Notice: $Av = \lambda v$ is in same direction as v



$Av = \lambda v$

Domain

Codomain

Must be same dimension

$A = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$ v, λ

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda = 6$$

$$Av = \lambda v \qquad Av = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 5 \\ 4 \end{bmatrix} \cdot 1 = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\lambda v = 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

OBSERVE: EIGENVECTORS ASSOCIATED w/ SAME λ

NOT UNIQUE

$$Av = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda v \qquad A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A(2v) = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \lambda(2v) \qquad A\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

ANY SCALAR MULTIPLE OF EIG VEC IS
ANOTHER EIG VEC w/ SAME λ

ICA 2:

Below are all the eigenvectors / eigenvalues of A. Match each eigenvector to its corresponding eigenvalue.

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$$

AND $\lambda = -1$

$$v = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$\lambda = -1$

$$Av = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} (-5) + \begin{bmatrix} 5 \\ 4 \end{bmatrix} 2 = \begin{bmatrix} -5 \\ -10 \end{bmatrix} + \begin{bmatrix} 10 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \lambda v$$

FINDING EIGENVALUES

WANT: METHOD OF FINDING λ, v PAIRS FROM A

IDENTITY

HAVE: $Av = \lambda v \rightarrow Av = \lambda I v$

$$Av - \lambda I v = \vec{0}$$

$$(A - \lambda I)v = \vec{0}$$

λ IS AN EIGVALUE WHEN THERE IS NON-ZERO COMBINATION OF COL OF $(A - \lambda I)$ EQUAL TO $\vec{0}$

WANT λ WHICH GIVE $\det(A - \lambda I) = 0$

FIND EIGENVALUES OF $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 5 \\ 2 & 4-\lambda \end{bmatrix}$$

$$0 = \mathrm{DET}(A - \lambda I) = (1-\lambda)(4-\lambda) - 2 \cdot 5$$

$$= 4 - 4\lambda - \lambda + \lambda^2 - 10$$

$$= \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1)$$

$\lambda = 6$ OR $\lambda = -1$

EVERY $N \times N$ MATRIX HAS N EIGENVALUES

SINCE

EVERY POLYNOMIAL OF ORDER N HAS N ROOTS

FIND CORRESPONDING EIGENVECTOR $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$

$\lambda = -1$ or 6

$(A - \lambda I)v = \vec{0}$

$v = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} 10 \\ -4 \end{bmatrix}$

$$\left(\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) v = \vec{0}$$

$$\begin{bmatrix} 2 & 5 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \vec{0} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} 10 + \begin{bmatrix} 5 \\ 5 \end{bmatrix} -4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Finding an eigenvalue of square matrix:

- Find roots of "characteristic polynomial"

SOLVE $\det(A - \lambda I) = 0$

Finding an eigenvector associated with eigenvalue:

SOLVE $(A - \lambda I)v = \vec{0}$

ICA FIND ALL EIGENVECTOR / EIGENVALUE PAIRS

OF $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

$\lambda = 8$ OR $\lambda = -2$

$0 = \mathrm{DET}(A - \lambda I) = \mathrm{DET}\left(\begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix}\right)$

$= (7-\lambda)(-1-\lambda) - 3 \cdot 3$

$= -7 + \lambda - 7\lambda + \lambda^2 - 9 = \lambda^2 - 6\lambda - 16$

$= (\lambda - 8)(\lambda + 2)$

$$A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \qquad \lambda = -2$$

$$(A - \lambda I)v = \vec{0}$$

$$\left(\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right) V = \vec{0}$$

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 \\ 3 \end{bmatrix} v_0 + \begin{bmatrix} 3 \\ 1 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 9 & 3 & 0 \end{array}\right]$$

$$\begin{array}{cc} v_0 & v_1 \end{array}$$

$$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \quad 3v_0 + 1v_1 = 0$$

$$v = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$\lambda = 8 \quad v_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

EIG VEC ORTHOGONAL

$\lambda = -2 \quad v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\times \pi$

$\times -1$

$\lambda_1 = -2 \quad v_1 = \begin{bmatrix} \pi \\ -3\pi \end{bmatrix}$

$\lambda_2 = -2 \quad v_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$Av = 0v = 0$

$\rightarrow$ V IS LIN COMBO COL A EQUAL 0

$\rightarrow$ A IS NOT INVERTIBLE

Numpy

How does the Magic Trick work?

$$\begin{bmatrix} x_0' \\ x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} .2 & .3 & .4 \\ .2 & .6 & .5 \\ .6 & .1 & .1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

Pop After Move

Pop Before Move

Insight: Write x as a linear combo of eig vecs of A

$$x = c_0 v_0 + c_1 v_1 + c_2 v_2$$

$$Av = \lambda v$$

$$\underline{A^{100}x} = A^{100}\left(c_0 v_0 + c_1 v_1 + c_2 v_2\right)$$

$$= c_0 \underline{A^{100} v_0} + c_1 \underline{A^{100} v_1} + c_2 \underline{A^{100} v_2}$$

$$= c_0 \lambda_0^{100} v_0 + c_1 \lambda_1^{100} v_1 + c_2 \lambda_2^{100} v_2$$

IS THIS GOOD FOR ANYTHING BESIDES "MAGIC"?

"MATRIX DIAGONALIZATION" IS COMPUTATIONALLY EFFICIENT

$$A^{100}x = c_0 \lambda_0^{100} v_0 + c_1 \lambda_1^{100} v_1 + c_2 \lambda_2^{100} v_2$$

ONE EXPONENT COMPUTE

100 MATRIX MULTIPLICATIONS