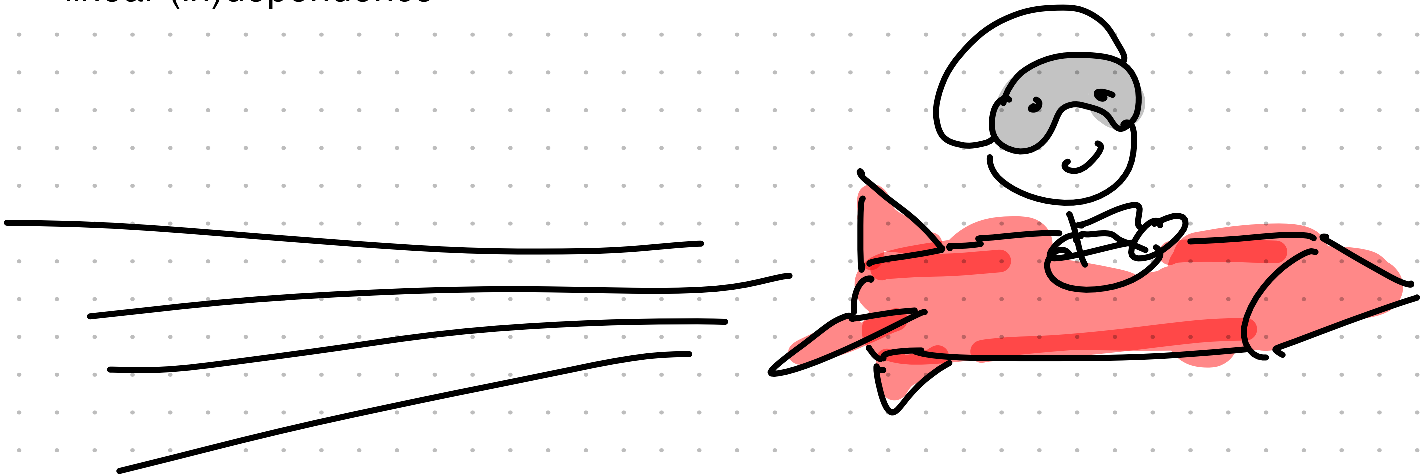
CS 2810 Day 7
Feb 8 2022

Admin: ICA in notes

Describing a set of vectors:
- span
- linear (in)dependence



An astronaut is spinning in outer space and needs to stop before they get dizzy! Their spaceship needs impulse b= [10, -11, 0]^T to stop rotating, what control signals x_0, x_1 should they use with their boosters a_0, a_1, ... to stop?

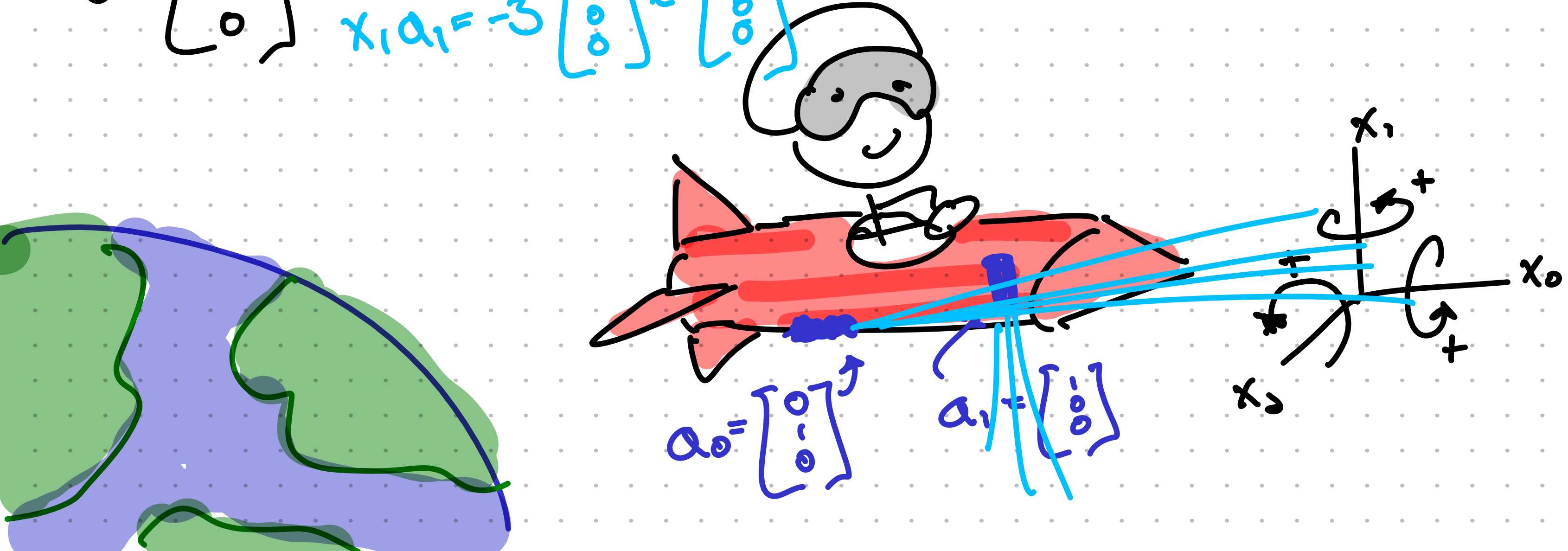
$$b = \begin{bmatrix} 10 \\ -11 \\ 0 \end{bmatrix}$$

$x_1 = -3$

$$x_1 a_1 = -3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

$x_0 = 10$

$$x_0 a_0 = 10 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$



$$x_0 a_0 + x_1 a_1 = 10\begin{bmatrix}0\\1\\0\end{bmatrix} - 3\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-3\\10\\0\end{bmatrix}$$

$$\begin{bmatrix}\text{[illegible]}\\\text{[illegible]}\\\text{[illegible]}\end{bmatrix} = -11\begin{bmatrix}0\\1\\0\end{bmatrix} + 10\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\-11\\0\end{bmatrix} + \begin{bmatrix}10\\0\\0\end{bmatrix} = \begin{bmatrix}10\\-11\\0\end{bmatrix}$$

ICA 1: An astronaut is spinning in outer space and needs to stop before they get dizzy! Their spaceship needs impulse \vec{b} = [b_0, b_1, b_2] to stop rotating.

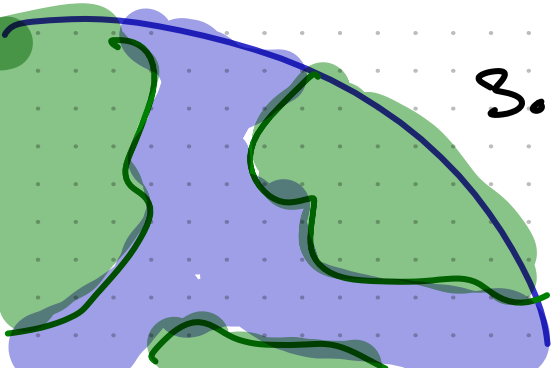
- Is there always a control signal x_0, x_1 which produces the needed impulse, for any \vec{b}?
- If not, which impulses, b, can be generated from boosters a_0, a_1?
- What boosters would you need to add to ensure that the rocket can produce any impulse?

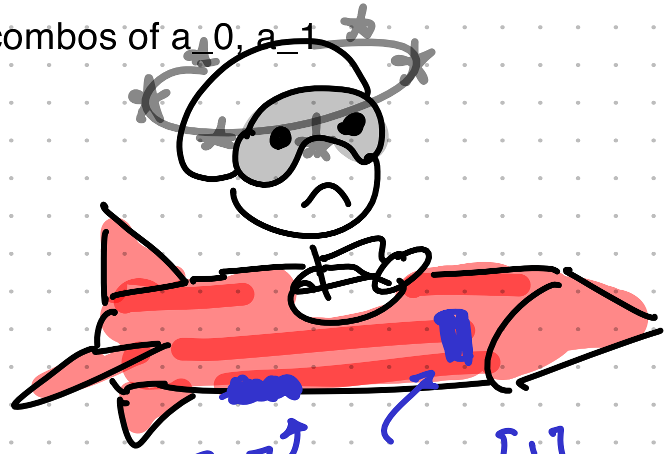
1. No! there exists some impulses b, which are not linear combos of a_0, a_1

2. [r_0, r_1, 0] where r_i in Real

$$\begin{bmatrix} \approx \\ \approx \\ 0 \end{bmatrix}$$

3. ADD $a_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$





$a_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Span [and its rocket ship connection]

The span of a set of vectors a_0, a_1, a_2, ... is the set of all vectors which can be written as a linear combination

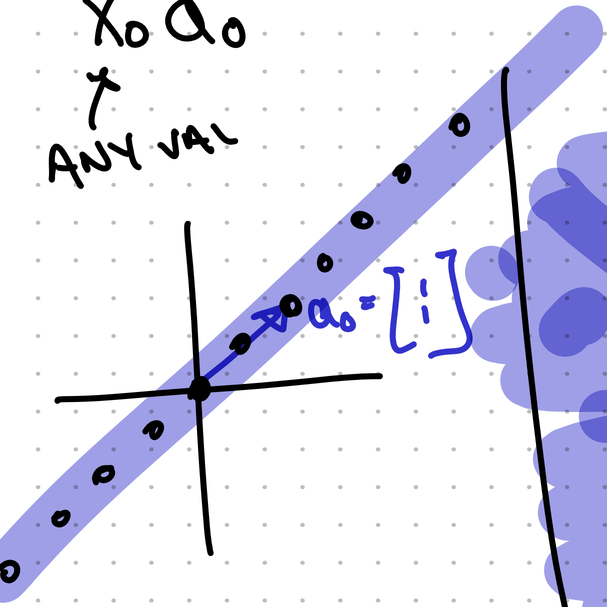
$$\text{SPAN}(\vec{a}_0\ \vec{a}_1\ \vec{a}_2) = \{x_0\vec{a}_0 + x_1\vec{a}_1 + \dots \mid x_i \in \mathbb{R}\}$$

[The span of boosters a_0, a_1, a_2 are all the impulses that they could possibly create]

ICA 2: SHADE THE SPAN OF EACH SET OF VECTORS

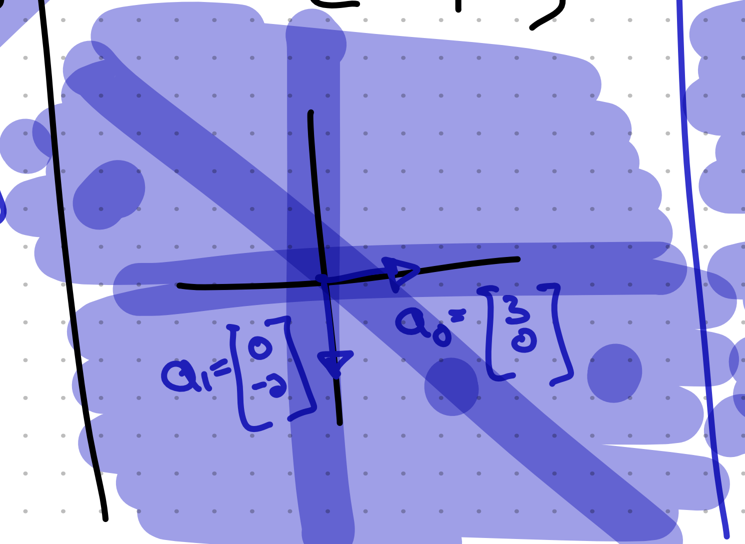
$x_0 a_0$

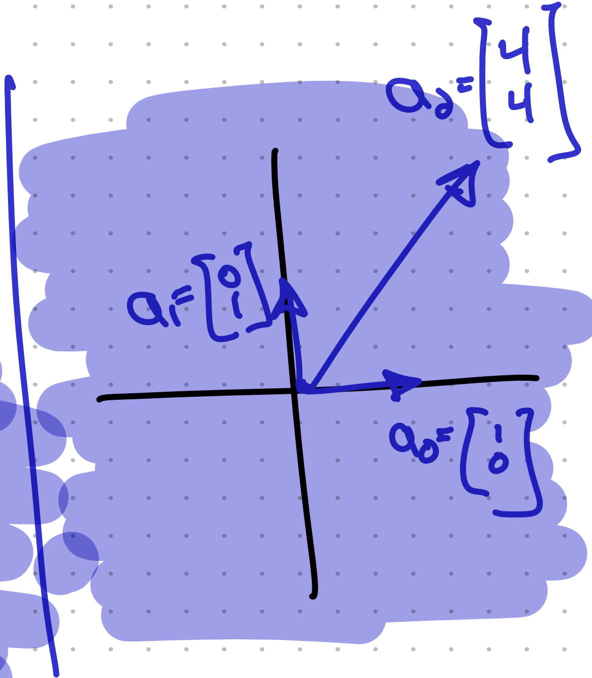
ANY VAL

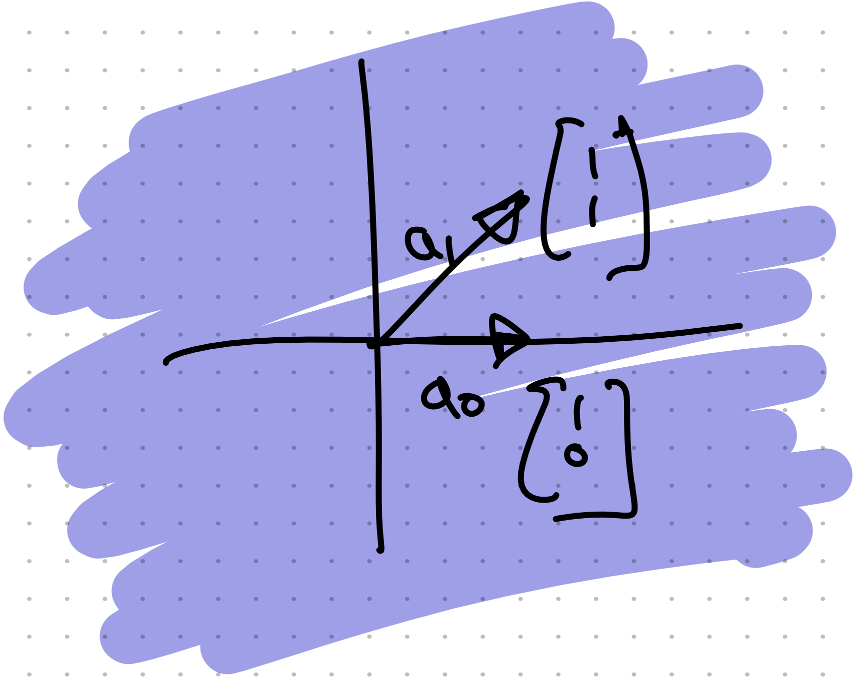


$$x_0\begin{bmatrix}3\\0\end{bmatrix}+x_1\begin{bmatrix}0\\-2\end{bmatrix}=\begin{bmatrix}*\\*\end{bmatrix}$$

$$\left[\begin{array}{cc|c}3 & 0 & *\\0 & -2 & *\end{array}\right]$$



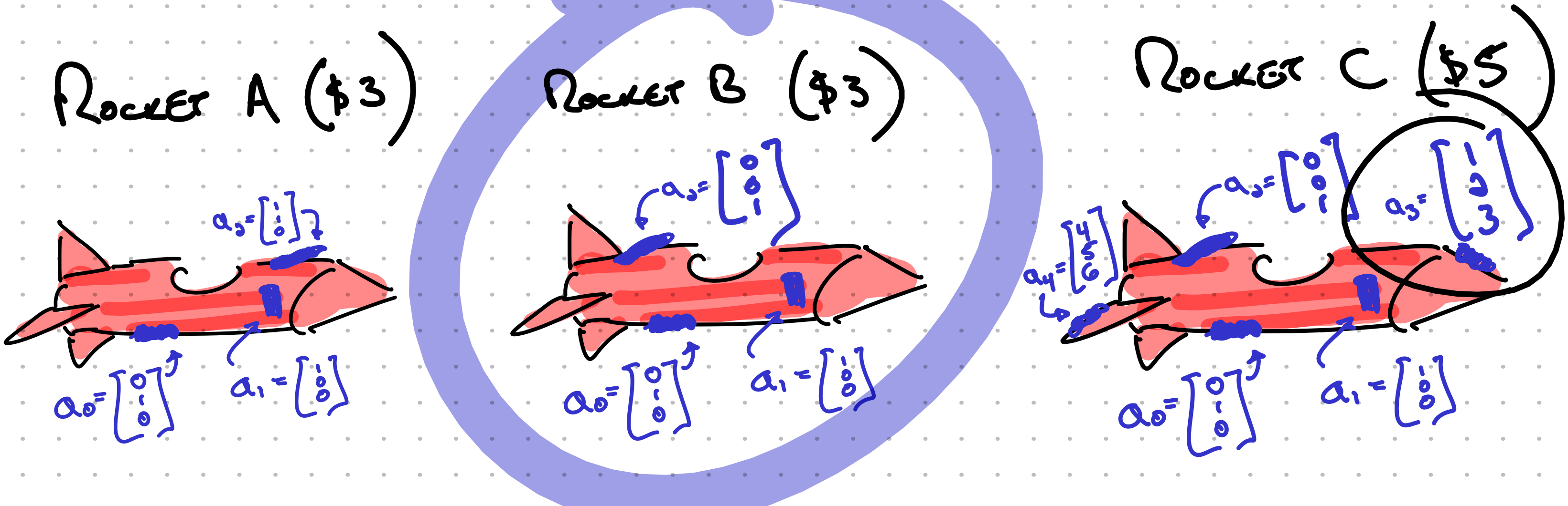




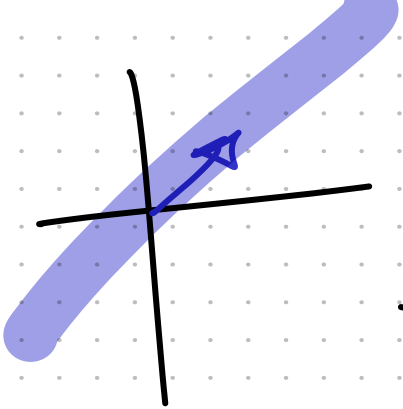
$$\begin{bmatrix}111\\70\end{bmatrix} = 70\begin{bmatrix}1\\1\end{bmatrix} + 41\begin{bmatrix}1\\0\end{bmatrix}$$

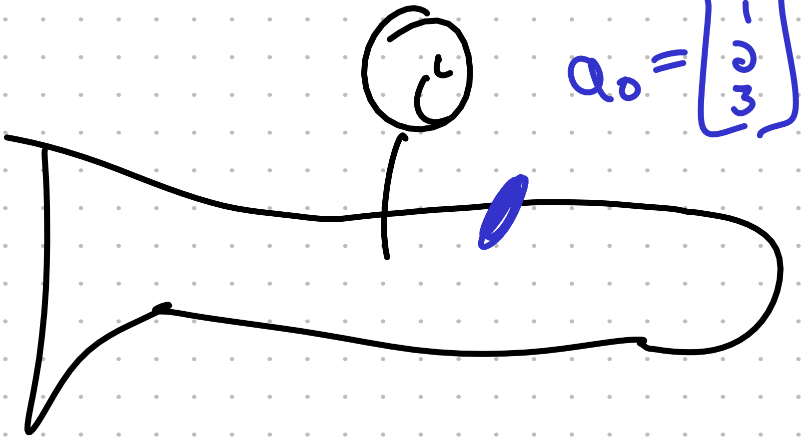
ICA 3: Which of the rockets below is capable of producing any rotation \vec{b} = [b_0, b_1, b_2]^T while costing the least amount of money?
What conditions must a_0, a_1, a_2, ... meet so that the rocket can produce any rotation?

Rocket B can produce all rotations and it doesn't have any "wasted" boosters which don't do anything new



$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$





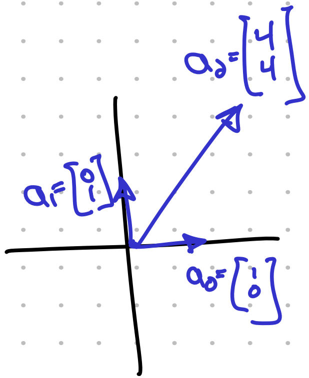
$$a_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Linear dependence (definition 1 of 2)

We say that a set of vectors a_0, a_1, a_2, ... is linearly dependent if some vector can be written as a linear combination of the others:

THERE EXISTS $x_i \in \mathbb{R}$ WITH $a_0 = x_1 a_1 + x_2 a_2 + \ldots$

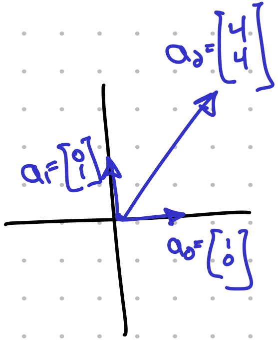
[Linearly dependent boosters are wasteful ... they produce an impulse which could've been created by the other boosters]

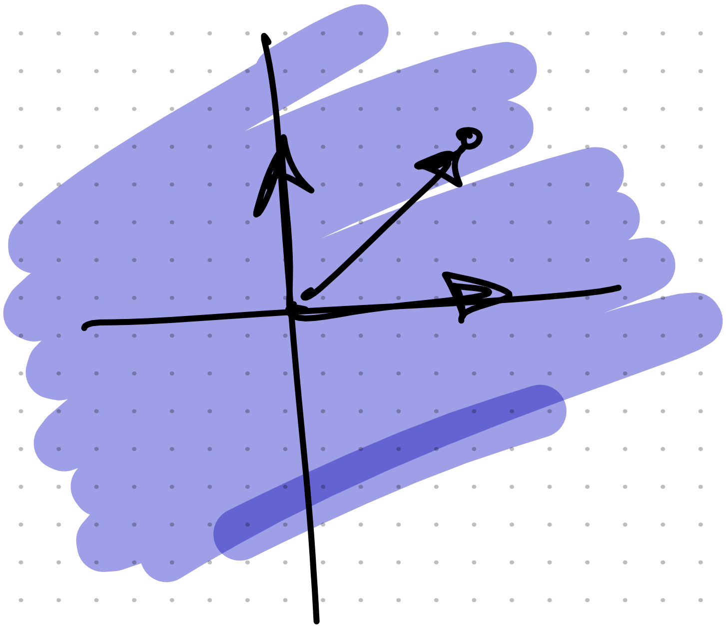


Linear dependence (definition 2 of 2)

We say that a set of vectors a_0, a_1, a_2, ... is linearly dependent if there exists some scalars x_0, x_1, ... (not all equal to zero) with:

$$\vec{0} = x_0 \vec{a}_0 + x_1 \vec{a}_1 + \dots$$





TESTING FOR LINEAR DEPENDENCE

ARE $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ LINEARLY DEPENDENT?

<u>GOAL</u>: FIND ALL x_0 x_1 x_2 WITH

$$\vec{0} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} x_2$$

IF ANY NON ZERO SOLUTIONS EXIST $\rightarrow$ LINEAR DEPENDENCE

TESTING FOR LINEAR DEPENDENCE

GOAL: FIND ALL $x_0\ x_1\ x_2$ WITH

$$\vec{0} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 2 & 1 & 4 \\ 2 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{r_1' = r_1 - r_0} \left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{r_2' = r_2 + r_1} \left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

row of zeros -> many solutions exist -> some non-zero solution exists -> vectors linearly dependent

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} y + \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} z = \begin{bmatrix} 2 & 1 & 4 \\ 2 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$0 = 2x + 1y + 4z$$

$$0 = 2x + 0y + 2z$$

$$0 = 0x + 1y + 2z$$

Testing for Linear Dependence

Are $\begin{bmatrix} | \\ a_0 \\ | \end{bmatrix}, \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix}, \begin{bmatrix} | \\ a_2 \\ | \end{bmatrix}$ linearly dependent?

$$\underset{1}{\begin{bmatrix} | & | & | \\ a_0 & a_1 & a_2 \\ | & | & | \end{bmatrix}} \xrightarrow{\;2\;} \underset{3}{\begin{bmatrix} \text{Reduced Row Echelon Form} \end{bmatrix}}$$

1. Build matrix
   each row is a vector from given

2. Row reduce to RREF

3. Inspect RREF...
If RREF has zero row:
   vecs are linearly dependent
else:
   vecs are linearly independent

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

ICA 4

Determine if the following set of vectors is linearly independent

$$\begin{bmatrix}1\\-2\\0\end{bmatrix} \quad \begin{bmatrix}4\\0\\8\end{bmatrix} \quad \begin{bmatrix}3\\-1\\5\end{bmatrix}$$

```
matt@matt-yoga1:~$ python3
Python 3.8.10 (default, Nov 26 2021, 20:14:08)
[GCC 9.3.0] on linux
Type "help", "copyright", "credits" or "license" for more i
>>> import sympy
>>> x = sympy.Matrix([[1, 4, 3], [-2, 0, -1], [0, 8, 5]])
>>> x
Matrix([
[ 1, 4,  3],
[-2, 0, -1],
[ 0, 8,  5]])
>>> x.rref()
(Matrix([
[1, 0, 1/2],
[0, 1, 5/8],
[0, 0,   0]]), (0, 1))
>>>
```

RREF

?

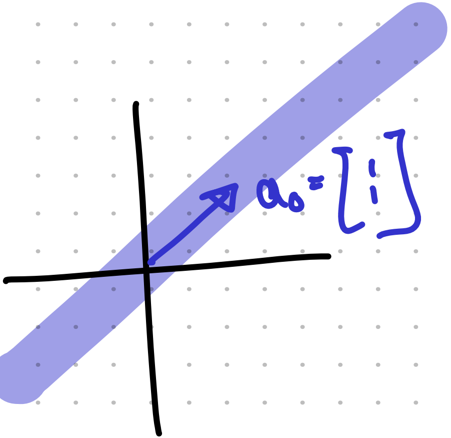
ZERO ROW

→ MANY SOL

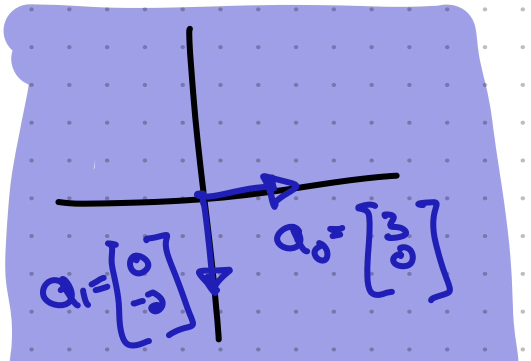
→ SOME NON-ZERO SOL

→ LINEAR DEPENDENCE

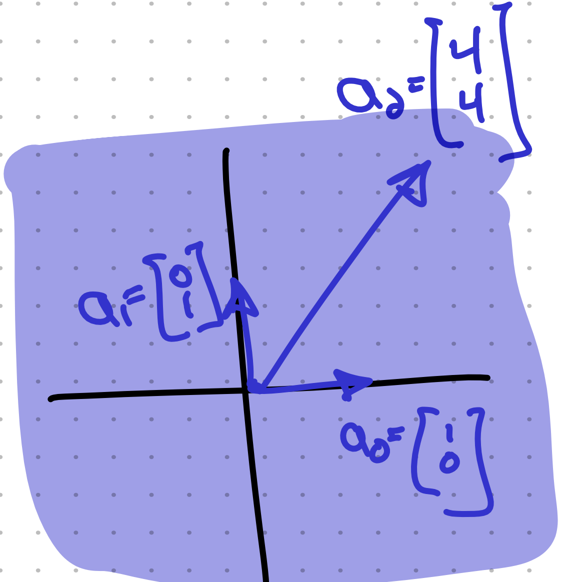
The span of N linearly independent vectors is an N dimensional space
0d space: point
1d space: line
2d space: plane
3d space: we live in a 3d space, ...



1 linear indep vec
->1d span



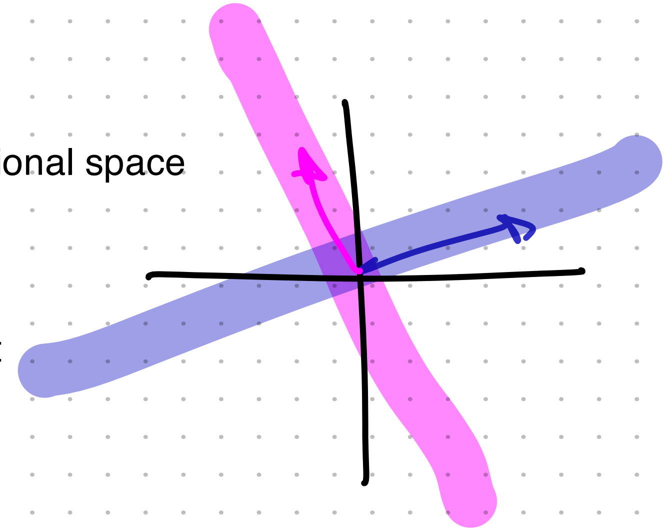
2 linear indep vecs
-> 2d span



3 linear dependent vecs
-> span is not 3 dimensional

Some final, helpful facts to remember:

1. The span of N vectors is never more than an N dimensional space



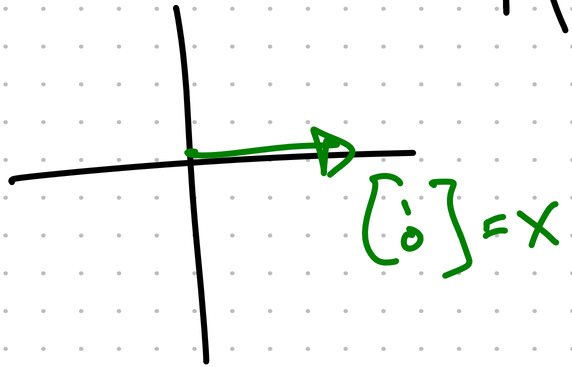
2. N+1 or more vectors of length N are linearly dependent

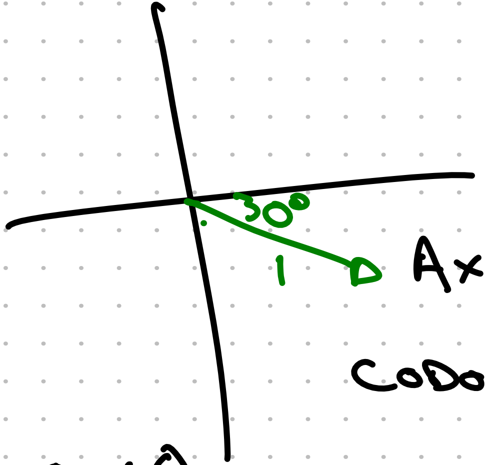
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

3 VECTORS OF LENGTH 2

FIND 2x2 MATRIX A WHICH ROTATES A VECTOR 30 DEG CLOCKWISE AROUND ORIGIN

$$A = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$





$$Ax = \begin{bmatrix} \cos 30 \\ -\sin 30 \end{bmatrix}$$

CODOMAIN

$$\begin{bmatrix} ? \\ ? \end{bmatrix} = Ax = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot a_0 + 0\, a_1 = a_0$$