CS800
11/13-Fri

Admin

- prog mtgs next week
- PS7 due 11:59pm
- PS8 at 11/14, due 11/20
- 12/4 exam 3

Agenda

1. Summations
2. Summation formulas
3. Intro to induction

1. Summations

Sequence - ordered list of numeric values

↳ addition of ordered list of numeric values

(when sequence is infinite, adding the values => series)

when sequence is finite, that is a partial sum

$\sum$ sigma - add the values

$$\sum_{\text{starting point}}^{\text{stop point}}$$

(ex) $\sum_{i=1}^{7} i$ the sum as i goes from 1 to 7 (inclusive) of i

$i = 1, 2, 3, 4, 5, 6, 7$

by definition...

$$\sum_{i=1}^{7} i = 1+2+3+4+5+6+7 = 28$$

what we care about...

$$\sum_{i=1}^{7} i = \boxed{?}$$ what specific value?

why we care...

fast: single computation

slow: repetition / loop

last time - sequences

a_k = value at position k

slow way :\

- compute a_1
- compute a_2
- compute a_3
- ...
- compute a_{k-1}
- compute a_k

fast way :)

- compute a_k directly
- all we need is k :)

generalize the summation $\sum_{i=1}^{n} i = 1+2+3+\ldots+n$ (by definition)

meet in the middle $\sum_{i=1}^{100} i = \boxed{?}$ $1+2+3+\ldots+100 = ?$

Gauss

Sum of first 100 positive integers
5050

add ↳ $1 + 2 + 3 + \dots + 99 + 100$
$100 + 99 + 98 + \dots + 2 + 1$
$101 + 101 + 101 \dots + 101 + 101$

How many 101s do we have? 100 !!

$100 \cdot 101$

but, we have the sequence twice!

$$\frac{100 \cdot 101}{2} = 50 \cdot 101 = 5050$$

generalized sum $\sum_{i=1}^{n} i$

- write seq, and write in reverse
- sum the values
- how many values do we have, what are they?

$1 + 2 + 3 + \dots + (n-1) + n$
$n + (n-1) + (n-2) + \dots + 2 + 1$
$(n+1), (n+1), (n+1)$ $(n+1), (n+1)$ n values each is $n+1$

$$\sum_{i=1}^{n} i = \frac{(n)(n+1)}{2}$$

First n pos integers

2. Summation Formulas

↳ if we are adding together any arithmetic sequence, there is a more general formula

ex $\sum_{i=1}^{?} ? = 1 + 3 + 5 + \dots + 97 + 99$

↳ arithmetic sequence!
$a = 1 \quad d = 2$

$a + (i-1)d$ (i is position)
$1 + (i-1)2$
$1 + 2i - 2$
$2i - 1$

$i=1, \ 1$
$i=2, \ 3$
$i=3, \ 5$
$i=4, \ 7$

$$\sum_{i=1}^{50} (2i-1) = 1 + 3 + 5 + \dots + 97 + 99$$

$\sum_{i=1}^{1}(2i-1) = 1 = 1^2$ $\sum_{i=1}^{2}(2i-1) = 1+3 = 4 = 2^2$ $\sum_{i=1}^{3}(2i-1) = 1+3+5 = 9 = 3^2$

$\sum_{i=1}^{4}(2i-1) = 1+3+5+7 = 16 = 4^2$ $\sum_{i=1}^{5}(2i-1) = 1+3+5+7+9 = 25 = 5^2$

$\sum_{i=1}^{50}(2i-1) = 1+3+\dots+99 = 50^2 = 2500$

any arithmetic sequence value of partial sum $\frac{\#\text{terms} \cdot (\text{first} + \text{last})}{2}$

(ex) first 100 pos integers

# terms	100
first	1
last	100

$\frac{100 \cdot (100+1)}{2} = \frac{100 \cdot 101}{2} = 5050$

(ex) odd positive integers $1, \dots, 99$

# terms	50
first	1
last	99

$\frac{50 \cdot (1+99)}{2} = \frac{50 \cdot 100}{2} = 2500$

any geometric sequence value of partial sum $\frac{a \cdot r^n - a}{r-1}$

Sequence: 1, 2, 4, 8, 16, 32, 64, 128

add-binate sense

$2^8 - 1 = 255$

rep in binary

1	1
2	10
4	100
8	1000
16	10000
⋮	

(add together, get all 1s)

formula

$a = 1 \quad r = 2 \quad n = 8$

$\frac{1 \cdot 2^8 - 1}{(2-1)} = 2^8 - 1 = 255$

(ex) formula for sum of the first n terms of $2, 4, 6, 8, 10, \ldots$

- arithmetic sequence

$$\frac{(\#\text{terms})(\text{first} + \text{last})}{2}$$

#terms: n

first: 2

last: whatever is at position n

$$a_n = a + (n-1)d$$
$$= 2 + (n-1)2$$
$$= 2 + 2n - 2$$
$$= 2n$$

$$\frac{(n)(2+2n)}{2}$$
$$= \frac{2n + 2n^2}{2}$$
$$= n^2 + n$$

3. Mathematical Induction

Proof techniques in CS1800
- truth tables
- logical equivalence laws
- set equality laws
- induction ★★

Categories where it applies:
- value of summations
- inequalities
- program correctness
- structural

Start with: $\sum_{i=1}^{n} i = \frac{(n)(n+1)}{2}$

Qs about the proof...
- convincing? (1-5)
- why predicate and logic statement?
- what is purpose of implication?
- when do we say "by inductive hypothesis"

Predicate $P(x)$ states that $\sum_{i=1}^{x} i = \frac{x(x+1)}{2}$ generalization of logic statement

Logic Statement $\forall n \in \mathbb{Z}^+ \, P(n)$ add quantifier - now it's something we can prove!

Base case:

$P(1)$ $\sum_{i=1}^{1} i = \frac{(1)(1+1)}{2} = \frac{2}{2} = 1$ this is trivially true

assume true for $\sum_{i=1}^{k} i$, show for $\sum_{i=1}^{k+1} i$

Inductive step: We want to show $P(k) \Rightarrow P(k+1)$

Inductive hypothesis: $P(k)$ is true: $\sum_{i=1}^{k} i = 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$

We want to show $P(k+1)$ $\sum_{i=1}^{k+1} i = 1 + 2 + 3 + \ldots + k + 1 = \frac{(k+1)(k+2)}{2}$

Start with the definition of $\sum_{i=1}^{k+1} i$ and express the summation including the term preceding $k+1$:

$\sum_{i=1}^{k+1} i = 1 + 2 + 3 + \ldots + k + 1 =$ $1 + 2 + 3 + \ldots + k + (k+1)$ ← apply P(k)

$= \frac{k(k+1)}{2} + (k+1)$; by inductive hypothesis

$= \frac{k(k+1)}{2} + 2(k+1)/2$; Multiply second term by 2/2

$= \frac{k(k+1)+2(k+1)}{2}$; Add fractions

$= \frac{k^2+k+2k+2}{2}$; Multiply through

$= \frac{k^2+3k+2}{2}$; Rearrange

$= \frac{(k+1)(k+2)}{2}$; Polynomial. Done!