CS1800
11/10 - Tues.

Admin

- next wk - proj mtgs
- PS7 due 11/13 11:59pm

Agenda

1. sequences
2. sequence types
3. deriving formulas

1. sequences — this week seqs → sums → induction

↳ discrete structure
help us discover + characterize patterns

Sequence – ordered list of numbers
mapping from $\mathbb{Z}^+$ to another set
might be infinite

notation – $\{a_n\}$ represents a sequence → order matters!

$\{a_1, a_2, \ldots, a_n\}$ finite sequence with n elements

$\{a_1, a_2, a_3, \ldots\}$ infinite sequence

a_k the k^{th} element in the sequence

$\mathbb{Z}^+$	$\longrightarrow$	$\{a_n\}$
1		2
2		4
3		8
4		16
5		32
⋮		⋮

Option #2 better - why?
- more flexible, pick any k
- faster! compute one thing

sequence $\{a_n\} = \{2, 4, 8, 16, 32, \ldots\}$ → position in sequence

- infinite
- $a_1 = 2 = 2^1$
  $a_2 = 4 = 2^2$
  $a_3 = 8 = 2^3$
  $a_4 = 16 = 2^4$
  $a_5 = 32 = 2^5$
  ...

$a_{10} = ?$

#1 double each time (pattern) keep going:
$a_6 = 64$ $a_7 = 128$ $a_8 = 256$
$a_9 = 512$ $a_{10} = 1024$

#2 compute value from your position in sequence. $2^{10} = 1024$

- $a_k = 2^k$ → compute the value at arbitrary position

We care about with sequences:

- given a formula for a_k, give first few terms *
- given a formula for a_k, compute the value at position k !!
- given the first few terms:
  - what is formula for a_k?

* given a formula, give first 5 terms

$\{a_n\}$ given by $a_k = \frac{k}{k+1}$

$a_1 = 1/2$
$a_2 = 2/3$
$a_3 = 3/4$
$a_4 = 4/5$
$a_5 = 5/6$

!! given a formula, compute the value at k

$\{a_n\}$ given by $a_k = (-1)^k$

$a_{308} = 1$

● given first few terms, what is formula?

$\{a_n\}$ with terms $\{7, 9, 11, 13, 15, \ldots\}$ $a_k = 5 + 2k$

$a_1 = 5 + 2 \cdot 1 = 7$
$a_2 = 5 + 2 \cdot 2 = 9$
$a_3 = 5 + 2 \cdot 3 = 11$
...

2. Types of Sequences → easier to derive a formula once you know the sequence type

arithmetic

- add same value over & over

$a, a+d, a+2d, a+3d, \ldots$

a = initial value

d = common difference

ex: $\{7, 9, 11, 13, 15, \ldots\}$

$a = 7 \quad d = 2$

$a_k = 7 + (k-1)d$

geometric

- multiply same value over & over

$a, ar, ar^2, ar^3, \ldots$

a = initial value

r = common ratio

ex: $\{2, 4, 8, 16, 32, \ldots\}$

$a = 2 \quad r = 2$

$a_k = 2^k$

quadratic

- derived from quadratic formula $a \cdot k^2 + bk + c$
- k = pos in sequence
- save for a, b, c

$a_1 = 7 + (1-1)\cdot 2 = 7$

$a_2 = 7 + (2-1)\cdot 2 = 9$

$a_3 = 7 + (3-1)\cdot 2 = 11$

...

a_k for arithmetic sequence

$$a_k = a + (k-1)\cdot d$$

a_k for geometric seq:

$$a_k = a \cdot r^{k-1}$$

$a = 2 \quad r = 2$

$a_1 = 2 \cdot r^0 = 2$

$a_2 = 2 \cdot r^1 = 2 \cdot 2 = 4$

$a_3 = 2 \cdot r^2 = 2 \cdot 4 = 8$

...

3. Deriving a formula → context: want to give formula for a_k
what is value at position k?

- what type of sequence? 1. is it arithmetic?
first level diffs are the same

$a_k = a + (k-1)d$

(ex) 7, 9, 11, 13, 15, ...

2 2 2 2 (first level diffs)

2. is it geometric?
ratios are the same

$a_k = a \cdot r^{(k-1)}$

(ex) 2, 4, 8, 16, 32, 64, ...

2 2 2 2 2 ratios

3. is it quadratic
second level diffs are the same

$a_k = a \cdot k^2 + b \cdot k + c$

solve for a, b, c

(ex) 1, 3, 6, 10, 15, 21, ...

2 3 4 5 6 (not same)

1 1 1 1 (quadratic!

Given first few terms:

- is it arith, geo, quad?
- what is a_k?

(#1) $\{a_n\} = 6, 2, 2/3, 2/9, 2/27, \ldots$

- geometric sequence! $a = 6$ $r = 1/3$

$$a_k = a \cdot r^{(k-1)} = 6 \cdot \left(\frac{1}{3}\right)^{(k-1)}$$

(ex) $\{a_n\} = 4, 1, -2, -5, \ldots$

- arithmetic sequence $a = 4$ $d = -3$

$$a_k = a + (k-1)d = 4 + (k-1)(-3)$$

(ex) $\{a_n\} = 9, 18, 31, 48, \ldots$

- quadratic! first diffs: $9, 13, 17, \ldots$
  second diffs: $4, 4, \ldots$

$a_k = a \cdot k^2 + b \cdot k + c$ (solve for a, b, c)

solving for three variables

$a_1 = a + b + c = 9$

$a_2 = 4a + 2b + c = 18$

$a_3 = 9a + 3b + c = 31$

isolate variables

$a_2 - a_1$

$$\begin{array}{r} 4a + 2b + c = 18 \\ -\ a + b + c = 9 \\ \hline 3a + b = 9 \end{array}$$

$b = 9 - 3a$

$a_3 - a_2$

$$\begin{array}{r} 9a + 3b + c = 31 \\ -\ 4a + 2b + c = 18 \\ \hline 5a + b = 13 \end{array}$$

plug in $b = 9 - 3a$

$5a + 9 - 3a = 13$

$5a - 3a = 4$

$2a = 4$

$\boxed{a = 2}$

$b = 9 - 3a$

$= 9 - 3 \cdot 2$

$= 9 - 6$

$= 3$

$\boxed{b = 3}$

$a + b + c = 9$

$2 + 3 + c = 9$

$c = 9 - 5$

$= 4$

$\boxed{c = 4}$

$$a_k = a \cdot k^2 + b \cdot k + c$$
$$= 2 \cdot k^2 + 3k + 4$$

our <u>other</u> quadratic seq: 1, 3, 6, 10, 15, 21, ...

$a_k = \frac{(k)(k+1)}{2}$

- Solve for a_k, knowing only k
- closed form

(our option #2 from before! fast, efficient :))

2 3 4 5 6 ⟶ 1st level diffs

$\boxed{a_k = a_{k-1} + k \quad a_1 = 1}$ recursive formula + base case

$a_2 = a_1 + 2$

$a_3 = a_2 + 3$

$a_4 = a_3 + 4$

...

(recursive version: less fast, maybe more intuitive)