CS1802
Fall 2026
Handout: Sequences And Summations

A **sequence** is an enumerated collection of objects -- a little like a set, because we have a bunch of objects, but now *order matters*. We use the notation a_n to denote the *n*th term in a sequence. The first term can be term #0 or term #1; in the module videos we started at 1 so we'll mostly stick with that.

In general, we might be interested in...

- Listing out the first several terms of a sequence,
- calculating the value of a_n for an arbitrary *n*, and/or
- summing together the first *n* terms.

Arithmetic Sequence

The terms in an arithmetic sequence follow a pattern of the form $a, a + d, a + 2d, a + 3d\ldots$ We call *a* the **initial value** and *d* the **common difference**.

Example -- the arithmetic sequence $3, 8, 13, 18, 23, 28\ldots$ has initial value 3 and common difference 5.

To find the *k*th term in an arithmetic sequence, for any *k*, use the formula:

$$a_k = a + (k - 1) \cdot d$$

Geometric Sequence

The terms in an arithmetic sequence follow a pattern of the form $a, a \cdot r, a \cdot r^2, a \cdot r^3\ldots$ We call a the **initial value** and *r*the **common ratio.**

Example -- the geometric sequence: $3, -6, 12, -24, 48\ldots$ has initial value 3 and common ratio -2.

To find the *k*th term a_k in a geometric sequence, for any *n*, use the formula:

$$a_k = a_1 \cdot r^{k-1}$$

Quadratic Sequence

The terms in a quadratic sequence increase or decrease at a constant rate. It's not as simple as arithmetic, where we're always adding on the common difference *d*. Instead, when we look at the differences between values in a quadratic sequence, the **differences make an arithmetic sequence.**

Example -- if we look at the sequence: $1, 4, 9, 16, 25\ldots$, it's clearly not arithmetic. The differences between the numbers are $3, 5, 7, 9, \ldots$. But! Look at the differences as a sequence of their own: $3, 5, 7, 9\ldots$ is arithmetic.

We can figure out the **second difference from here** -- the difference of the differences. Here, it's $2, 2, 2, 2, \ldots$, telling us that the numbers in the original sequence increase at a constant rate. Therefore, quadratic!

To find the *k*th term a_k in a quadratic sequence, for any *k*, use the formula:

$$a_k = ak^2 + bk + c$$

And then we need the values for *a, b,* and *c* to actually apply the formula.

If you have the first few terms of a quadratic sequence, you can figure out the values of *a, b*, and *c*, and then you can use those to find the value of any arbitrary term. Figuring out the values of *a, b,* and *c* is good ol' algebra.

For the example $1, 4, 9, 16, 25\ldots$, we can just jump to the end because it's quite simple. $a_k = ak^2 + bk + c$ where $a = 1, b = 0, c = 0$.

Arithmetic + Geometric Sums

A summation is the addition of the terms of **any** sequence; it can apply to any kind of ordered list of numbers. It also turns out, if the sequence you're summing together is arithmetic or geometric, there are super useful formulas we can easily apply to figure out the value of a summation.

Anytime we're adding together the first *n* terms of a sequence, the result is some specific value, usually part of a pattern. And if we know what that value turns out to be, we want to show that it's true when we add together the first term of a sequence, the first two terms of a sequence, the first three terms of a sequence, and so on. Doesn't that sound like a job for proof by induction? Heck yes it does!

Let's start with a simple arithmetic sequence: 1, 2 ,3, 4, ... If we add together the first *n* terms of the sequence, we get $\frac{n(n+1)}{2}$. Now let's prove it!

Predicate $P(x)$ states that $\sum_{i=1}^{x} i = \frac{x(x+1)}{2}$

Logic Statement $\forall n \in \mathbb{Z}^+ P(n)$

Base case:

$P(1)$ $\sum_{i=1}^{1} i = \frac{(1)(1+1)}{2} = \frac{2}{2} = 1$ this is trivially true

Inductive step: We want to show $P(k) \Rightarrow P(k+1)$

Inductive hypothesis: $P(k)$ is true: $\sum_{i=1}^{k} i = 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$

We want to show $P(k+1)$ $\sum_{i=1}^{k+1} i = 1 + 2 + 3 + \ldots + k + 1 = \frac{(k+1)(k+2)}{2}$

Start with the definition of $\sum_{i=1}^{k+1} i$ and express the summation including the term preceding $k+1$:

$$\sum_{i=1}^{k+1} i = 1 + 2 + 3 + \ldots + k + 1 = \quad 1 + 2 + 3 + \ldots + k + (k+1)$$

$$= \quad \frac{k(k+1)}{2} + (k+1) \qquad \text{; by inductive hypothesis}$$

$$= \quad \frac{k(k+1)}{2} + 2(k+1)/2 \qquad \text{; Multiply second term by 2/2}$$

$$= \quad \frac{k(k+1)+2(k+1)}{2} \qquad \text{; Add fractions}$$

$$= \quad \frac{k^2+k+2k+2}{2} \qquad \text{; Multiply through}$$

$$= \quad \frac{k^2+3k+2}{2} \qquad \text{; Rearrange}$$

$$= \quad \frac{(k+1)(k+2)}{2} \qquad \text{; Polynomial. Done!}$$

Of course the sequence 1, 2, 3, ..., x is arithmetic, with initial value 1 and common difference 1. A generalization of the formula $\frac{(x)(x+1)}{2}$ applies to **any** arithmetic series, not just that specific one.

Extra Fun: General Summation Formulas

Value of an arithmetic partial sum: $\frac{\#terms \cdot (first + last)}{2}$

For example, let's go back to adding together the first 100 positive integers:

- Number of terms: 100
- First term: 1
- Last term: 100

Formula: $\frac{\#terms \cdot (first + last)}{2} = \frac{(100)(101)}{2} = 5050$ **same as before it must be working!!**

Value of a geometric partial sum: $\frac{ar^n - a}{r - 1}$

This one is a little more complex than the arithmetic one, so let's break it down to see how we got here. Recall how we defined a geometric sequence: $a,\ ar,\ ar^2,\ ar^3,\ \ldots$ where *r* is the common ratio and *a* is the initial value.

It follows that a geometric partial sum is the sum of the first *n* terms of an geometric sequence

$$\sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \ldots + ar^{n-1}$$

We can get to the formula by taking this summation, and then also multiplying it by *r* so that we have two formulas going at once. Let's label the summations to make things easier on ourselves, and then, if we subtract one from the other, we'll be in excellent shape:

$$S \qquad a + ar + ar^2 + \ldots + ar^{n-1}$$

$$r \cdot S \quad ar + ar^2 + ar^3 \ \ldots \ + ar^n$$

$$r \cdot S - S = (ar + ar^2 + ar^3 \ \ldots \ + ar^n) - (a + ar + ar^2 + \ldots + ar^{n-1})$$

$$= ar^{n+1} - a$$

Almost done, just one last tweak here. We have:

$$r \cdot S - S = ar^n - a$$

But remember that *S* was the label for the initial summation, the one we're trying to come up with a formula for. Let's take this equation and solve for *S,* and then we're done!

$$r \cdot S - S = ar^n - a$$

$$S(r - 1) = ar^n - a$$

$$S = (ar^n - a)/(r - 1)$$ **that's what we started with so we're good!!!!**

For example, let's say we have this geometric sequence: 1, 2, 4, 8, 16, 32, 64, 128.

- a: 1
- r: 2
- n: 8

Formula: $\frac{ar^n - a}{r-1} = \frac{1 \cdot 2^8 - 1}{(2-1)} = \frac{255}{1} = 255$

Sanity check, add them by hand: $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 = 255$ **it's working!!!!**