CS1800
Fall 2026
Handout: Basic Probabilities

Recall that the Pigeonhole Principle, which makes guarantees about a certain outcome. Now we're still in the general realm of counting, but instead we're focused on the **likelihood** of a certain outcome. Of course, you can take a whole course in probability and statistics; in Discrete Structures we focus on the fundamental concepts.

To start with probabilities, we first need to define an **experiment** - a procedure that can be infinitely repeated and has a well-defined set of possible outcomes.

Did you catch that last part? An experiment has a well-defined **SET** of possible outcomes -- yes, our old friend! Let's turn our attention to sets as we go through a few probability-related vocab words:

- **Sample Space (S):** Set of all possible outcomes of an experiment
- **Event (E):** Subset of the sample space

For now, we'll assume all outcomes in the sample space are equally likely. We'll define $Pr(E)$: the probability of event E , given by $\frac{|E|}{|S|}$.

Rolling One Die
For example, suppose our experiment is rolling a six-sided die. The sample space is all possible outcomes:

$$S = \{1, 2, 3, 4, 5, 6\}$$

An event E could be any subset of S. Let's consider an event E that captures rolling a number less than 3:

$$E = \{1, 2\}$$

We can now compute the probability: $Pr(E) = \frac{|E|}{|S|} = \frac{2}{6} = \frac{1}{3}$.

If we're interested in multiple, distinct events, the **sum rule** comes back into our lives. Let's say we care about rolling a number less than 3 or greater than 5. We'd have:

$$E_1 = \{1, 2\}, E_2 = \{6\}$$

The probability of rolling a number less than 3 OR greater than 5 means we can apply the sum rule:

$$Pr(E_1) + Pr(E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{3}{6} = 1/2$$

Note that in this example, the two events do not have any elements in common. The same approach still works even if the outcomes overlap, we just need to be careful about those union and cardinality operations, because a set contains distinct elements. Let's say we care about rolling a number less than 3 or rolling an even number. We'd have:

$E_1 = \{1, 2\}, E_2 = \{2, 4, 6\}$

The probability of rolling a number less than 3 OR an even number:

$Pr(E_1) + Pr(E_2) = \frac{|E_1 \cup E_2|}{|S|} + \frac{4}{6} = 2/3$

Exactly and At Least/At Most Probabilities

A bit string is any string made up only of "1" and "0". Let's see how to compute how likely a certain kind of bit string is. Suppose a bit string of length 5 is randomly generated by a robot, with every outcome equally likely.

How many bit strings are there in total? $2^5 = 32$. That gives us our sample space, $|S| = 32$.

- What is the probability that a bit string contains 5 zeroes?
  $E = \{00000\}$
  $|E| = 1$
  $Pr(E) = \frac{1}{32} = .0312$

- What is the probability that a bit string ends with a 1?
  $E = \{x \mid x\ is\ a\ bitstring\ of\ the\ form\ \ ----\ 1\}$
  $|E| = 16$
  $Pr(E) = \frac{16}{32} = .5$

- What is the probability that a bit string contains **exactly** three consecutive zeroes?
  $E = \{00011, 00010, 10001, 11000, 01000\}$
  $|E| = 5$
  $Pr(E) = \frac{5}{32} = .156$

- What is the probability that a bit string contains **at least** three consecutive zeroes?
  $E = \{x \mid x\ is\ a\ bitstring\ of\ the\ form\ 000\ --,\ -\ 000\ -,\ --\ 000\}$
  - Start with three zeroes: 000_ _ There are $2^2 = 4$ of these.
  - Three zeroes in the middle: _000 _ There are $2^2 = 4$ of these, but 00000 and 00001 are already counted above, so there are 2 of these.
  - End with three zeroes: _ _000. There are $2^2 = 4$ of these, but 00000 and 10000 are already counted above, so there are 2 of these.

  $|E| = 4 + 2 + 2 = 8$
  $Pr(E) = \frac{8}{32} = .25$

Joint Probability

Joint probability helps us understand how likely it is for two or more events to happen at the same time. For examples in this section, let's say we have two drawers of socks. Drawer A has 10 pink socks and 5 orange socks; Drawer B has 16 pink socks and 8 orange socks. Therefore the probability of randomly drawing a pink sock from either drawer is 2/3 and the probability of randomly drawing an orange sock from either drawer is 1/3.

When events E and F are **independent**, meaning that the occurrence of one event does not impact the other, we use our old friend the product rule to compute the probability of **E and F:**

$$Pr(E \cap F) = Pr(E) \cdot Pr(F)$$

Example: Let E be the event of drawing a pink sock from Drawer A. Let F be the event of drawing an orange sock from Drawer B. These two events are independent, and so we have the probability of drawing pink from A AND orange from B is:
$Pr(E \cap F) = Pr(E) \cdot Pr(F) = 2/3 \cdot 1/3 = 2/9$

Let's explore all the probabilities involved when we draw a sock from Drawer A and then a sock from Drawer B. It can help to organize probabilities of multiple events in a **joint probability table**.

	Drawer A- Pink	**Drawer A - Orange**	**Pr(Drawer B)**
Drawer B - Pink	$2/3 \cdot 2/3 = 4/9$	$1/3 \cdot 2/3 = 2/9$	$4/9 + 2/9 = 2/3$
Drawer B - Orange	$2/3 \cdot 1/3 = 2/9$	$1/3 \cdot 1/3 = 1/9$	$2/9 + 1/9 = 1/3$
Pr(Drawer A)	$4/9 + 2/9 = 2/3$	$2/9 + 1/9 = 1/3$	

Each of the four inner cells shows the probability of a particular combination of results from the two drawers (pink then pink = 4/9, orange then pink = 2/9, etc); these probabilities are the joint distribution.

Then the rightmost column (*PR(Drawer B)*) and bottom-most row (*Pr(Drawer A)*) are called the **marginal probabilities**. Each of these values tells us about Drawer A or Drawer B independent of other things going on. We use the **sum rule** to compute these marginal probabilities, because the probability of getting a pink from Drawer B happens when we select pink-pink OR orange-pink.

In the rightmost column, we see what happens with Drawer B no matter what happened with drawer A:

- PR(pink from Drawer B) = 2/3, because we could select pink-pink OR orange-pink.
- PR(orange from Drawer B) = 1/3, because we could select pink-orange OR orange-orange.

In the bottom-most row, we see what happens with Drawer A no matter what happened with drawer B:

- PR(pink from Drawer A) = 2/3, because we could select pink-pink OR pink-orange.
- PR(orange from Drawer A) = 1/3, because we could select orange-pink OR orange-orange.