CS1800
Fall 2026
Handout: Mathematical Induction

Induction is a proof method used often in computer science and mathematics. It's at its most useful when you're attempting to prove that something is true for every positive integer, or for every natural number.

Recall that we can use a predicate to generalize a logic statement, where $P(x)$means that the statement P applies to the input x. With induction, we are **usually** trying to prove a predicate $P$for all positive integers: $\forall n \in Z+, P(n)$

If you had all the time and patience in the universe, you could go through one by one and prove that $P(1)$ is true, $P(2)$is true, $P(3)$is true, $P(4)$ is true, $P(5)$ is true, and omg I'm already so bored of writing these. Induction gives us a shortcut to prove all of these at once.

There are four parts to a proof by mathematical induction:

- **Predicate** -- the starting point
- **Logic statement** -- Using quantifiers, turn the predicate into a logic statement; this is what we want to prove.
- **Base case** -- the smallest value of x for which you want to show $P(x)$is true. This should be straightforward.
- **Inductive step** -- the heart of the proof! Start with an *inductive hypothesis*, where you assume $P(k)$is true .Then you prove an implication: $P(k) \Rightarrow P(k+1)$.

Short Mathematical Detour
Just a quick sidestep here. Sometimes in proofs like this, we need to expand polynomials and apply exponents, so here is a little refresher in case some time has passed since you've seen those things:

Expanding Polynomials (you should know these two pretty well, but for anything more complex it's no big deal to look it up!)

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Exponent Rules

$a^{-b} = \frac{1}{a^b}$ ex: $3^{-3} = \frac{1}{3^3} = 1/27$

$(ab)^c = a^c \cdot b^c$ ex: $(4 \cdot 3)^3 = 4^3 \cdot 3^3 = 1728$

$a^b \cdot a^c = a^{b+c}$ ex: $2^3 \cdot 2^5 = 2^8 = 256$

$(a^b)^c = a^{b+c}$ ex: $(2^3)^5 = 2^{15} = 32786$

Induction Example #1 -- Summation

Predicate $P(x)$ states that $\sum_{i=1}^{x} i = \frac{x(x+1)}{2}$

Logic Statement $\forall n \in \mathbb{Z}^{+} P(n)$

Base case:

$P(1)$ $\sum_{i=1}^{1} i = \frac{(1)(1+1)}{2} = \frac{2}{2} = 1$ this is trivially true

Inductive step: We want to show $P(k) \Rightarrow P(k+1)$

Inductive hypothesis: $P(k)$ is true: $\sum_{i=1}^{k} i = 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$

We want to show $P(k+1)$ $\sum_{i=1}^{k+1} i = 1 + 2 + 3 + \ldots + k + 1 = \frac{(k+1)(k+2)}{2}$

Start with the definition of $\sum_{i=1}^{k+1} i$ and express the summation including the term preceding $k+1$:

$\sum_{i=1}^{k+1} i = 1 + 2 + 3 + \ldots + k + 1 =$ $1 + 2 + 3 + \ldots + k + (k+1)$

$= \frac{k(k+1)}{2} + (k+1)$ // by inductive hypothesis

$= \frac{k(k+1)}{2} + 2(k+1)/2$ // Multiply second term by 2/2

$= \frac{k(k+1)+2(k+1)}{2}$ // Add fractions

$= \frac{k^2+k+2k+2}{2}$ // Multiply through

$= \frac{k^2+3k+2}{2}$ // Rearrange

$= \frac{(k+1)(k+2)}{2}$ // Polynomial. Done!

Induction Example #2 -- Divisibility

Predicate $P(x)\ states\ that\ 6^x + 4\ is\ a\ multiple\ of\ 5$

Logic statement $\forall n \in Z^+ P(n)$

Base case

$P(1)$ $6^1 + 4$ is a multiple of 5

$6^1 + 4 = 10$ True!

Inductive step $P(k) \Rightarrow P(k+1)$

Inductive Hypothesis $P(k)$, assumed true: $(6^k + 4)$ is a multiple of 5.

$P(k+1)$states that $(6^{k+1} + 4)$ is a multiple of 5, but we don't know yet if it's true. This is what we want to prove. Start by plugging in *k*+1 to get the expression we care about, and then do things that don't modify the value of the expression.

$(6^{k+1} + 4)$ = $6^k \cdot 6^1 + 4$ // exponent law $a^b a^c = a^{b+c}$

= $6^k \cdot (5+1) + 4$ // 6 = 5+1. We do this because 5 is useful!

= $6^k \cdot 5 + 6^k \cdot 1 + 4$ //Distribute the 6^k

$6^k \cdot 1 + 4$is a multiple of 5, by our Inductive Hypothesis.

$6^k \cdot 5$ is multiple by 5, by definition of multiplication

And we know that the sum of two multiples of 5 is itself a multiple of 5.

Therefore, we have shown that $6^{n+1} + 4$ is a multiple of 5 whenever *n* is a positive integer. This completes the proof.

Induction Example #3 -- 3 cents and 5 cents

Claim *P*(*n*) states that we can make *n*-cents of change using only 3-cent and 5-cent pieces.

We want to show that, $\forall n \in \mathbb{Z}^{+},\ n > 7,\ P(n)$

More formally, we want to show that we can express n, for any $n > 7$, in terms of some number $t$of three-cent pieces and and some number $f$of five-cent pieces, i.e., $n = 3t + 5f$ with $t \geq 0$ and $f \geq 0$.

Base case:

$P(8)$ $\quad 8 = 3t + 5f$ $\quad$ True when $t = 1$and $f = 1$.

Inductive step: We want to show $P(k) \Rightarrow P(k + 1)$

Inductive hypothesis: $P(k)$ is true: $k = 3t + 5f$

Here, we show that if we have $k$cents made of 3-cent and 5-cent pieces, we can make $k + 1$cents.

$k = 3t + 5f$ $\quad$ by inductive hypothesis

Case one: We have no five-cent coins (e.g., $f = 0$).
Now our inductive hypothesis is: $k = 3t$. We'll show that we can get $k + 1$cents by subtracting 3 three-cent pieces and adding two five-cent pieces.

$k + 1 = 3t + 1$	// add one to both sides
$= 3t + (10 - 9)$	// express 1 as 10 - 9
$= 3(t - 3) + 10$	// rearrange
$= 3(t - 3) + 5 \cdot 2$	// express 10 as 5 times 2

Done! Now we have $k + 1$cents with $t - 3$three-cent pieces and 2 five-cent pieces.

Case two: We have at least one five-cent coin
Now our inductive hypothesis is: $k = 3t + 5f$. We'll show that we can get $k + 1$cents by adding 2 three-cent pieces and subtracting one five-cent piece.

$k + 1 = 3t + 5f + 1$	// add one to both sides
$= 3t + 5f + 6 - 5$	// express 1 as 6 - 5
$= 3(t + 2) + 5(f - 1)$	// rearrange

Done! Now we have $k + 1$cents with $t + 2$three-cent pieces and $f - 1$ five-cent pieces.

Since we know $P(8)$is true, and because we've shown the implication $P(k) \Rightarrow P(k+1)$, therefore $P(n)$ is true for all $n > 7$. This completes the proof.