Nov 12

Admin

- Project meetings next week.
- PS7 due Friday the 13th.

Agenda

- Seq. Wrapup
- Series & Summations
- Induction.

Sequences

↳ Pattern recognition: given some terms → find k^{th} term. formula.

↳ Next few terms → continue sequence

↳ Given k^{th} term formula → write out a few terms.

Arithmetic seq vs. Geometric seq.

Arithmetic seq

- ∃ common difference d b/w consecutive terms

$$a_{k+1} - a_k = d \quad \forall k$$

- First/initial value : a
- k^{th} term $a_k = a + (k-1)d$

Geometric seq.

- ∃ common ratio r b/w consecutive terms

$$\frac{a_{k+1}}{a_k} = r \quad \forall k$$

- First/initial value : a
- k^{th} term $a_k = a \cdot r^{(k-1)}$

Quadratic Sequence

eg: 1, 4, 9, 16, 25, ... $\rightarrow$ $a_k = k^2$

In general $a_k = ak^2 + bk + c$

Test for Quadratic Seq

1, 4, 9, 16, 25 $\rightarrow$ quad seq $\rightsquigarrow$ $a_k = ak^2 + bk + c$

3, 5, 7, 9 $\rightarrow$ arith seq

2, 2, 2 $\rightarrow$ common diff.

$a_1 = a + b + c = 1$

$a_2 = 4a + 2b + c = 4$

$a_3 = 9a + 3b + c = 9$

$\rightarrow$ solve for a, b, c

you'll get $a = 1, b = c = 0$

$\therefore a_k = k^2$

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eg: 6, 15, 28, 45, 66, 91 $\rightarrow$ quad seq

9, 13, 17, 21, 25 $\rightarrow$ arith. seq

4, 4, 4, 4 $\rightarrow$ comm. diff

$a_k = ak^2 + bk + c$

$a_1 = 6 = a + b + c$ — ①

$a_2 = 15 = 4a + 2b + c$ — ②

$a_3 = 28 = 9a + 3b + c$ — ③

② − ①

$4a + 2b + c = 15$

$-\ a + b + c = 6$

$3a + b + 0c = 9$ $\hookrightarrow$ ④

③ − ②

$9a + 3b + c = 28$

$-\ 4a + 2b + c = 15$

$5a + b + 0 = 13$ ⑤

⑤ − ④

$5a + b = 13$

$-\ 3a + b = 9$

$2a = 4$

$a = 2$

use $a = 2$ in ④ $\Rightarrow 32 + b = 9$

$\Rightarrow b = 3$

use $a = 2, b = 3$ in ①

$3 + 2 + c = 6$

$\Rightarrow c = 1$

Verify that $a_k = 2k^2 + 3k + 1$

$a_1 = 2 + 3 + 1 = 6$ ✓

$a_2 = 8 + 6 + 1 = 15$ ✓

$a_3 = 18 + 9 + 1 = 28$ ✓

$a_4 = 32 + 12 + 1 = 45$ ✓

Examples: Determine Arith, Geom, Quad or None.
↳ find parameters.

① $-16, -14, -12, -10, \dots$

$-16 \xrightarrow{2} -14 \xrightarrow{2} -12 \xrightarrow{2} -10$ → comm diff

$a = -16, d = 2$

$\Rightarrow a_k = -16 + (k-1)2$

$= -18 + 2k$

② $\frac{1}{2}, \frac{-1}{4}, \frac{1}{8}, \frac{-1}{16}, \dots$

$\frac{1}{2} \xrightarrow{r=-1/2} \frac{-1}{4} \xrightarrow{r=-1/2} \frac{1}{8} \xrightarrow{r=-1/2} \frac{-1}{16}$ → comm ratio

$a = \frac{1}{2}, r = -\frac{1}{2}$

$a_k = \frac{1}{2} \cdot \left(-\frac{1}{2}\right)^{k-1}$

$= (-1)^{k-1} \cdot \frac{1}{2^k}$

③ $\underset{a_1}{10}, \underset{a_2}{9}, 7, 4, 0, \dots$ → quad.

$10 \xrightarrow{} 9 \xrightarrow{} 7 \xrightarrow{} 4 \xrightarrow{} 0$

$\xrightarrow{-1} \quad \xrightarrow{-2} \quad \xrightarrow{-3} \quad \xrightarrow{-4}$ → arith seq

$a_1 - a_0$

$\xrightarrow{-1} \quad \xrightarrow{-1} \quad \xrightarrow{-1} \quad \xrightarrow{-1}$ → comm diff = d

$a_k = -\frac{k^2}{2} + \frac{k}{2} + 10$

$a_k = ak^2 + bk + c$

$a = \frac{d}{2}$

$c = a_0$

$a_1 = a + b + c$

$a_1 = \frac{d}{2} + b + a_0$

$10 = \frac{-1}{2} + b + 10$

solve for b $\Rightarrow b = \frac{1}{2}$

④ $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ $\quad a_k = \frac{1}{k}$

Harmonic numbers

$H_n = \frac{1}{n}$

⑤ $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{42}, \ldots$ $\quad a_k = \frac{1}{k^2+k} = \frac{1}{k(k+1)}$

$\frac{1}{1\times2}, \frac{1}{2\times3}, \frac{1}{3\times4}, \frac{1}{4\times5}, \ldots$

0, 2, 6, 12, 20

2, 4, 6, 8

2, 2, 2

$a = 1 = d/2$

$c = 0 = a_0$

a_1

$1^2 + b = 2$

$b = 1$

⑥ $\frac{1}{1}, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$ $\quad a_k = \frac{1}{k!}$

⑦ $0, 1, 1, 2, 3, 5, 8, 13, \ldots$

$f_k = f_{k-1} + f_{k-2}$ → recursive definition.

$f_1 = 0, f_2 = 1$ → base case

OR. $f_0 = 0, f_1 = 1$ → most standard defⁿ.

⑧ $-4, 2, -2, 0, 2, 2, 4, \ldots$

$\begin{cases} b_k = b_{k-1} + b_{k-2} \\ b_1 = -4, b_2 = 2 \end{cases}$ full recursive defⁿ.

Series & Summations

$(a_n)_{n\in\mathbb{Z}^+}$ → Seq. ⇝ Series : $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots$ ← Sigma notation for summation.

partial sum of a series

$$\sum_{k=1}^{n} a_k$$

Last time we saw

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i = \sum_{k=1}^{n} k$$

Extend to formula for Arith. seq sum

$$\sum_{k=1}^{n} a + (k-1)d$$

$$[a+(1-1)d] + [a+(2-1)d] + [a+(3-1)d] + \cdots$$

$$= \left(\sum_{k=1}^{n} a\right) + \sum_{k=1}^{n} (k-1)d$$

$$= n\cdot a + d\left(\sum_{k=1}^{n} (k-1)\right)$$

$$= n\cdot a + d\left[\sum_{k=1}^{n} k + \sum_{k=1}^{n} (-1)\right]$$

$$= na + d\left[\frac{n(n+1)}{2} + (-1)\cdot n\right]$$

$$= \vdots$$

$$= \frac{n}{2}\left[a + \left(a+(n-1)d\right)\right]$$

$a+a+a+\cdots+a = n\cdot a$ (n times)

$= a[1+1+1+\cdots+1] = a\cdot n$ (n times)

the sum of n terms of an Arith. seq

$$\sum_{k=1}^{n} a+(k-1)d = \frac{\#\text{ of terms}}{2}\left(1^{st}\text{ term} + \text{last term}\right)$$

# of terms; first term; last term of Arith seq being summed