Nov 9

Admin

- PS7 out, due Friday the 13th
- Project Meetings next week.
  - ↳ signups due Nov 11th, Wed
- PS6 & Exam 2 grades tomorrow.

Agenda

- Sequences
- Summations
- Induction.

Sets vs. Sequences

- Order doesn't matter

  $\{1,2,3\} = \{2,1,3\}$

- Repetition not allowed

  $\{1,1,2\} = \{1,2\}$

- Order does matter

  $(1,2,3) \neq (2,1,3)$

- Repetitions allowed

  $(1,2,2,3,3,3)$

  $(1,2,2,3,3,3,4,4,4,4,\ldots)$

Most common ques about seq

- (Pattern Recognition)
- Next term
- n^{th} term
- Add terms of seq.

  Summation / series

Terminology

- Notation uses $(-,-,-,\dots) \to (a_1, a_2, a_3, \dots) \leftrightarrow (a_n)_{n \in \mathbb{Z}^+}$

- Elements / Terms of seq.

- Indexing into seq
  - 1st term, 2nd term, …, n^{th} term, etc.
  - can start indexing at 0.

- try to stick to one indexing.
- Be capable of switching

Example

① $(0, 1, 2, 3, \dots)$ → natural numbers from lowest to highest
- keep adding 1 to previous term
- "natural numbers" is not specific enough.) also fits

$(0, 2, 1, 4, 3, 6, 5, \dots)$

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② $(2, 4, 6, 8, \dots)$ → positive even #s in increasing order.
- +2 to previous term.

③ $(2, 3, 5, 7, \ldots)$

↳ $(2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots)$ ↳ primes in ascending order

$(2, 3, 5, 7, 9, 11, 13, 15, \ldots)$

↳ 2 followed by larger odd #s in inc. order

↳ first term is 2, second term is 3

n^{th} term is 2 + previous term.

↳ $(n > 2)$

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④ (a, e, i, o, u) ↳ vowels in alphabetic order.

⑤ (e, a, o, i, u) ↳ vowels in dec. freq. of occurrence in English

⑥ $(3, 3.1, 3.14, 3.141, 3.1415, \ldots)$

↳ rational approximations of π, with increasing precision.

⑦ $(1, 0.9, 0.99, 0.999, 0.9999, \ldots)$

$= (1, 1-0.1, 1-0.01, 1-0.001, \ldots)$

$= (1, 1-10^{-1}, 1-10^{-2}, 1-10^{-3}, \ldots)$

1st term is 1

$n > 1$, n^{th} term is $1 - 10^{-(n-1)}$

Can be helpful to index at 0.

$(1, 1-10^{-1}, 1-10^{-2}, \ldots)$ → 0^{th} term is 1

0^{th} term, 1^{st} term, 2^{nd} term

$n>0$, n^{th} term is $1-10^{-n}$

Arithmetic Seq. / Progression

eg: 2, 5, 8, 11, 14, ...

+3 +3 +3 +3

common difference $= d = a_{n+1} - a_n$

1^{st} term, 2^{nd}, 3^{rd}, 4^{th}, k^{th} term

$a, a+d, a+2d, a+3d, \ldots, a+(k-1)d$

↳ off by 1 is most common error } Be careful.

1^{st} term, initial value, starting term

equivalent ↳ $a_n = a + (n-1)d$ ↳ $(n \geq 1)$

Alternate formulation

$a_1 = a$

$a_n = a_{n-1} + d$ $(n>1)$

Quick eqs

Find a & d, k^{th} term for

$-7, 12, 17, 22, \ldots$ → $a=7, d=5$ $\quad a_k = 7+(k-1)5 = 2+5k$

$-4, 0, -4, -8, \ldots$ → $a=4, d=-4$ $\quad a_k = 4+(k-1)(-4) = 8-4k$

Find first 5 terms & 11^{th} term given

- $a=5, d=3$ → $5, 8, 11, 14, 17, \ldots$ $\quad 5+(11-1)\cdot 3 = 35$
- $a=-11, d=-2$ → $-11, -13, -15, -17, -19, \ldots$ $\quad -11+(11-1)(-2) = -31$

Geometric Seq / Progression

common ratio $= r = \frac{a_{n+1}}{a_n}$

eg. 2, 4, 8, 16, 32, ... (×2, ×2, ×2, ×2)

$a, \; ar, \; ar^2, \; ar^3, \; \dots, \; ar^{(k-1)}$

1st term, 2nd, 3rd, 4th, k^{th} term

a: initial value, etc

→ careful about ± 1.

equivalent

$$a_n = a \cdot r^{n-1} \quad (\text{for } n \geq 1)$$

Alternative form

$$a_1 = a$$
$$a_n = a_{n-1} \cdot r \quad (n > 1)$$

Quick eqs

Find a & r & k^{th} term

* $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots \rightarrow a = 1, \; r = \frac{1}{3}$; k^{th} term $= 1 \cdot \left(\frac{1}{3}\right)^{k-1} = 3^{1-k}$
* $1, 4, 16, 64, \dots \rightarrow a = 1, \; r = 4$; k^{th} term $= 1 \cdot (4)^{k-1} = 4^{k-1}$
* $1, 0.1, 0.01, 0.001, \dots \rightarrow a = 1, \; r = 10^{-1}$; k^{th} term $= 1 \cdot (10^{-1})^{k-1} = 10^{1-k}$

Find first 5 terms & 20^{th} term

- $a = 2, \; r = 4 \rightarrow 2, 8, 32, 128, 512, \dots$; $2 \cdot (4)^{20-1} = 2(2^2)^{19} = 2^{39}$
- $a = 1, \; r = -1 \rightarrow 1, -1, 1, -1, 1, \dots$; $1 \cdot (-1)^{20-1} = -1^{19} = -1$

Induction for Summation / Summation via Induction

eg $(1, 2, 3, 4, \ldots)$ → positive integers asc. order

↳ $a_i = i \quad (i \geq 1)$

Goal: $\underbrace{a_1 + a_2 + \cdots + a_n}_{\sum_{i=1}^{n} a_i} = \underbrace{1 + 2 + 3 + \cdots + n}_{\sum_{i=1}^{n} i}$

Define $P(n)$: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Claim → $\forall n \in \mathbb{Z}^+ : P(n)$ → Induction is great on these kind of proofs.

Proof:

Base case $P(1)$: $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$

$\Rightarrow 1 = 1$ True ✓

Inductive Hypothesis

Assume $P(n)$ is true $\forall n \in \{1, 2, \ldots, k\}$

↳ Using this we'll prove $P(k+1)$

$P(k+1): \sum_{i=1}^{k+1} i = \frac{(k+1)(k+1+1)}{2}$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

replace all n w/ $k+1$

$1 + 2 + 3 + \cdots + (k-1) + k + (k+1)$

is L.H.S of $P(k)$

$\left(\sum_{i=1}^{k} i\right) + (k+1)$

via Ind. Hyp.

$\frac{k(k+1)}{2} + k+1$

⋮ algebra

$= \frac{(k+1)(k+1+1)}{2}$ = R.H.S as required.

$\Rightarrow$ L.H.S = R.H.S

$\Rightarrow$ $P(k+1)$ is true

$\Rightarrow$ Inductive step done

$\Rightarrow$ Proof complete.