Nov 9

Admin

- PS7 out, due Friday the 13th
- Project Meetings next week.
  ↳ signups due Nov 11th, Wed
- PS6 & Exam 2 grades tomorrow.

Agenda

- Sequences
- Summations
- Induction.

Sets vs. Sequence

Sets

- Order doesn't matter
  $\{1,2,3\} = \{2,1,3\}$
- Repetition not allowed
  $\{1,1,2\} = \{1,2\}$

Sequence

- Order matters
  $(1,2,3) \neq (2,1,3)$
- Repetition is allowed.
  $(1,1,2,2,3,3)$
  $(1,1,2,2,3,3,4,4,\dots)$

Most common tasks w/ seq.

- (Pattern recognition)
- next element
- n^{th} element
- Add terms of sequence.
  Summation / series

Terminology

- Sequence notation uses $(\ ) \rightarrow (a_1, a_2, a_3, \ldots) = (a_n)_{n \in \mathbb{Z}^+}$
- elements / terms of the seq.
- Indexing into the seq.
  - ↳ 1st term, 2nd term, n^{th} term, etc.
  - ↳ Can also start indexing at 0.

- Try to stick to one form of indexing.
- Be capable of switching indexing.

Examples

① $(0, 1, 2, 3, \ldots)$ → Natural numbers in increasing order.

↳ pattern is +1 to previous term

↳ Natural numbers

↳ not fully describing the seq.

↳ also fits e.g. $(2, 1, 4, 3, 6, 5, \ldots)$

② $(2, 4, 6, 8, \ldots)$ → positive even numbers in ascending order.

③ (a, e, i, o, u) → vowels in alphabetical order.

a: 1st vowel, e: 2nd, i: 3rd, o: 4th, u: 5th

④ (e, a, o, i, u) → vowels in decreasing frequency of use in English.

⑤ $(2, 3, 5, 7, \ldots)$

↳ $(2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots)$

↳ prime #s in inc. order

$(2, 3, 5, 7, 9, 11, 13, 15, 17, \ldots)$

↳ 2 followed by odd #s > 2, in inc. order.

⑥ $(3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots)$

↳ rational approximations to π in inc. order of precision.

⑦ $(0.9, 0.99, 0.999, 0.9999, \ldots)$

$= (1-0.1, 1-0.01, 1-0.001, \ldots) = (1-10^{-1}, 1-10^{-2}, 1-10^{-3}, \ldots)$

↳ seq of $1-10^{-n}$, $n \geq 1$

Arithmetic Sequences

eg. 3, 7, 11, 15, 19, 23, ...

+4 +4 +4 +4 +4

1st 2nd 3rd k^{th} term

$a, a+d, a+2d, a+3d, \ldots, a+(k-1)d, \ldots$

Careful of off by-1 errors.

starting term
first term
initial value

both are formulations of an arithmetic seq.

(Arithmetic Progression)

Alternatively

$a_1 = a$

$a_n = a_{n-1} + d$

Quick eqs.

Find a & d for these arith. seqs.

- 6, 11, 16, 21, ... $a = 6, \; d = 5$

- 5, 1, -3, -7, ... $a = 5, \; d = -4$

Find first 5 terms & 20^{th} term of these arith. seqs.

- $a = 5, \; d = 3 \rightarrow 5, 8, 11, 14, 17, \ldots, 5+(20-1)\cdot 3$
$= 5 + 19 \cdot 3$
$= 62$

- $a = -10, \; d = -3 \rightarrow -10, -13, -16, -19, -22, \ldots, -10 + (20-1)(-3)$
$= -10 + 19(-3)$
$= -67$

Geometric Sequence / Progression

Eg. 2, 4, 8, 16, 32, 64, ---

×2 ×2 ×2 ×2 ×2

1st 2nd 3rd 4th ... k^{th}

$a, ar, ar^2, ar^3, \dots ar^{k-1}$

→ stay careful of ±1 errors

a: initial value

r: common ratio

Alternative formulation

$a_n = a$

$a_n = a_{n-1} \cdot r$

Examples

Find a & r & k^{th} term

- 3, 9, 27, 81, --- → $a=3, r=3$ → k^{th} term $= 3 \cdot 3^{k-1} = 3^k$
- 1, 4, 16, 64, --- → $a=1, r=4$ → k^{th} term $= 1 \cdot 4^{k-1} = 4^{k-1}$
- 1, 0.1, 0.01, 0.001, --- → $a=1, r=1/10$ → k^{th} term $= 1\left(\frac{1}{10}\right)^{k-1} = 10^{-(k-1)}$

Find first 5 terms & 11^{th} term of

- $a=2, r=4$ → 2, 8, 32, 128, 512 → $2 \cdot 4^{11-1} = 2 \cdot 4^{10} = 2 \cdot 2^{20} = \boxed{2^{21}}$
- $a=1, r=-1$ → 1, -1, 1, -1, 1, → $1(-1)^{11-1} = (-1)^{10} = 1$

Induction for Summation

eg. $(1, 2, 3, 4, 5, \ldots)$ → positive integers in inc. order

↳ Arith seq, $a=1, d=1$

$(a_n)_{n \in \mathbb{Z}^+}$

Goal: $1 + 2 + 3 + \ldots + a_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$

$= \sum_{i=1}^{n} i$

Claim: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ → call this predicate $P(n)$

↳ Want to prove: $\forall n \in \mathbb{Z}^+ : P(n)$ → Induction can apply.

Induction proof:

Base case: show that $P(1)$ ↳ $\sum_{i=1}^{n=1} i = 1$ ← L.H.S

$\frac{n(n+1)}{2} = \frac{1 \cdot (1+1)}{2} = 1$ ← R.H.S.

∴ $P(1)$ is true. Base case done.

Inductive Hypothesis:

Assume $\forall n \in \{1, 2, \ldots, k\}$ $P(n)$ is true.

Now we'll prove $P(k+1)$

$P(k+1)$: $\sum_{i=1}^{n=k+1} i = \frac{(k+1)(k+1+1)}{2}$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

substitute $n = k+1$

$1 + 2 + 3 + \cdots + (k-1) + k + k+1$

$\left(\sum_{i=1}^{k} i\right) + k+1$

(use Ind. Hyp

$\frac{k(k+1)}{2} + k+1$

$= \frac{(k+1)(k+2)}{2}$

$\therefore P(k+1)$ is true $\Rightarrow$ Ind. step is done

Proof finished.