Nov 2

Admin

- Exam 2 → Nov 4, Wed (in-class, 90 min)
  - 2's comp
  - sets
  - Counting
  - Prob (no cond. prob)
    - YES: Independence, Joint prob table
  - Reference sheet (1-sided)

Agenda

- Bayes Rule
- Expectation

Bayes Rule / Theorem

Cond prob: $Pr(B \mid A)$ → assume A has happened, what is prob of B now?
- "given"

"Symmetric form"

$$Pr(B \mid A) \cdot Pr(A) = Pr(A \mid B) \cdot Pr(B) = Pr(A \cap B)$$

- $Pr(B \mid A) \cdot Pr(A)$: A happened, then B
- $Pr(A \mid B) \cdot Pr(B)$: B happened, then A
- $Pr(A \cap B)$: Interpret as A & B both happened.

more useful form

$$Pr(B \mid A) = \frac{Pr(A \mid B) \cdot Pr(B)}{Pr(A)}$$

Eg. Urn Model / Balls in Bin

① Urn 1 | 99 white, 1 red

Urn 2 | 1 white, 99 red

Que 1. $P(R \mid U_1)$ ← red ball is drawn from Urn 1

$= 1/100$

Que 2 $P(U_1 \mid R) \xrightarrow{\text{Bayes Rule}} \dfrac{P(R \mid U_1) \cdot P(U_1)}{P(R)} = \dfrac{1/100 \cdot \cancel{100/200}}{\cancel{100/200}}$

$= 1/100$

Eg. Urn 1 | 30 red, 20 white

Urn 2 | 40 red, 20 white

$Pr(U_1 \mid R) = \dfrac{P(R \mid U_1) \cdot P(U_1)}{P(R)} = \dfrac{30/50 \cdot 50/110}{70/110}$

Eg. Urn 1 | 30 R, 20 W

Urn 2 | 40 R, 40 W

Urn 1 is used w/ prob 1/5

$Pr(U_1 \mid R) = \dfrac{P(R \mid U_1) \cdot P(U_1)}{P(R)} = \dfrac{30/50 \cdot 1/5}{??}$

$P(R) \rightarrow$ Pr of U_1 & then $R \rightarrow Pr(U_1 \cap R)$

$= Pr(R|U_1) Pr(U_1)$

OR

U_2 & then $R \rightarrow Pr(U_2 \cap R)$

$= Pr(R|U_2) \cdot Pr(U_2)$

$$P(R) = \underbrace{Pr(R|U_1) \cdot P(U_1) + P(R|U_2) \cdot P(U_2)}_{\text{is the denominator}}$$

$$Pr(U_1|R) = \frac{P(R|U_1) \cdot P(U_1)}{P(R)} = \frac{30/50 \cdot 1/5}{Pr(R|U_1) \cdot P(U_1) + P(R|U_2) \cdot P(U_2)}$$

Ref:

Urn 1	Urn 2
30 R	40 R
20 W	40 W

$$= \frac{3/5 \cdot 1/5}{3/5 \cdot 1/5 + 1/2 \cdot 4/5}$$

Example: Tuberculosis (Tb) → Tb has a test

$Pr(\text{Tb test is accurate}) = 95\%$

$Pr(\text{random person has Tb}) = 0.0005$

Via Bayes find $P(\text{has Tb} \mid \text{test +ve})$

+ve positive

-ve negative

$$P(Tb \mid +ve) = \frac{P(+ve \mid Tb) \cdot P_r(Tb)}{P_r(+ve)} = \frac{0.95 \times 0.0005}{P_r(+ve \mid Tb)\,P_r(Tb) + P_r(+ve \mid \overline{Tb}) \cdot P_r(\overline{Tb})}$$

$$= \frac{0.95 \times 0.0005}{0.95 \times 0.0005 + 0.05 \times 0.9995}$$

$$\approx 0.0094\ldots \approx 0.94\ \%$$

Last Bayes Eg of day

Supplier A : 4000 , 5% defects

B : 5000 , 4% defective

C : 4500 , 6% defective

Find $P_r(A \mid \text{defective}) = P_r(\text{product being from supplier A} \mid \text{product was defective})$

$$P(A \mid def) = \frac{P(def \mid A) \cdot P(A)}{P(def)} = \frac{0.05 + \frac{4000}{13500}}{P(def \mid A) \cdot P(A) + P(def \mid B) \cdot P(B) + P(def \mid C) \cdot P(C)}$$

$$= \frac{0.05 \cdot 40/135}{0.05 \times \frac{40}{135} + 0.04 \times \frac{50}{135} + 0.06 \times \frac{45}{135}}$$

Expectation

eg. fair coin toss → H you get $1
T you lose $2

Que. Should you play this game?

Ans. No. On average you lose money.
↳ $\frac{1+(-2)}{2} = -\frac{1}{2}$

What if unfair coin?
$P(H) = 0.6$ → earn $1
$P(T) = 0.4$ → lose $2

$\frac{60 \cdot (1) + 40(-2)}{100}$ [Tried 100 times]

$= \frac{60}{100} \cdot (1) + \frac{40}{100}(-2)$

weighted average ←

$= 0.6\,(1) + 0.4\,(-2)$

$= -\$0.2$

Formally: think experiment many, many times

$X \rightarrow$ random variable
↓
depends on outcome

$$X = \begin{cases} 1 & \text{if Heads} \\ -2 & \text{if Tails} \end{cases}$$

Expectation is the "average/expected" value of the random variable.

$$E[X] = P(S_1) \cdot X_1 + P(S_2) \cdot X_2 + \cdots$$

each possible outcome in Sample space S shows up once.

$$E[X] = \sum_{s_i \in S} p(s_i)\, x_i$$

Eg. Expected roll of 6-sided dice.

Soln: $E[\text{roll}] = \sum_{i \in \{1,2,\ldots,6\}} p(i) \cdot i$

$$= \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \cdots + \frac{1}{6} \cdot 6$$

$$= \frac{1+2+\cdots+6}{6} = \frac{21}{6} = \boxed{3.5}$$

Eg. Make a "fair game" involving rolling a 20-sided fair dice. ↳ $\{1, 2, \ldots, 20\}$

① Pay \$5 to roll 10 times → -\$½ cost/roll

② Odd rolls, you pay what you rolled.
Even rolls, you get what you rolled.

↳ $E[\text{roll}] = \frac{1}{20}\left[\underbrace{(-1)+2}_{1} + \underbrace{(-3)+4}_{1} + \underbrace{(-5)+6}_{1} + \cdots + (-19)+20\right]$

$$= \frac{1}{20} \cdot 10 = \frac{1}{2} \hookrightarrow \text{average earning per roll}.$$

Net effect: You earn/lose nothing.

Try yourself

3 coin tosses, coin is biased : $P_r(H) = 2/3$

value is H=1, T=-1, add values of all 3 tosses

What is the expected value of this?

eg HHH = 1+1+1 = 3 in value.

S_i	$P_r(S_i)$	X_i
TTT	$(1/3)^3$	-3
TTH	$(1/3)^2 \cdot 2/3$	-1
THT	$(1/3)^2 \cdot 2/3$	-1
THH	$(2/3)^2 \cdot 1/3$	1
HTT	$2/3 \cdot (1/3)^2$	-1
HTH	$(2/3)^2 \cdot 1/3$	1
HHT	$(2/3)^2 \cdot 1/3$	1
HHH	$(2/3)^3$	3

$$E[\text{3 tosses}] = \frac{1}{27}(-3) + \frac{2}{27}(-1) + \frac{2}{27}(-1) + \frac{4}{27}(1) + \frac{2}{27}(-1) + \frac{4}{27}(1) + \frac{4}{27}(1) + \frac{8}{27}(3)$$

$$= \frac{7 \cdot 3}{27} + \frac{12 - 6}{27} = \frac{21 + 6}{27} = \boxed{1}$$