Nov 2

Admin

- Exam 1 → Nov 5, Thu (in-class, 40 min)
  - ↳ 2's comp
  - ↳ sets
  - ↳ Counting
  - ↳ Prob (no cond. prob)
    - ↳ yes: Independence, Joint Prob Table.

Reference Sheet (1-side)

Agenda

- Bayes Rule
- Expectation

Bayes Rules / Theorem

Cond. prob : $Pr(B|A)$ → assume A has happened, what is prob of B now?

↳ "given"

"symmetric form"

$$Pr(B|A) \cdot Pr(A) = Pr(A|B) \cdot Pr(B) = Pr(A \cap B)$$

↳ Interpretation: A & B both happened.

$Pr(B|A) \cdot Pr(A)$: A happened, then B

$Pr(A|B) \cdot Pr(B)$: B happened, then A

more useful formulation

$$P(B|A) = \frac{P(A|B) \cdot Pr(B)}{Pr(A)}$$

Urn Model / Balls in Bins

Ques: $Pr(R \mid U_1)$ — red ball; drawn from Urn 1

① Urn 1 | Urn 2

Urn 1	Urn 2
99 white	1 white
1 red	99 red

$= \frac{1}{100}$

Que 2: $Pr(U_1 \mid R) \xrightarrow{\text{Bayes Rule}} \frac{Pr(R \mid U_1) \cdot Pr(U_1)}{Pr(R)} = \frac{1/100 \cdot 100/200}{100/200}$

$= \frac{1}{100}$

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② Tuberculosis → Tb has a test

$Pr(\text{Tb test is accurate}) = 95\%$

$Pr(\text{has Tb}) = 0.0005$

Goal: $Pr(\text{actually Tb} \mid \text{test says Tb})$

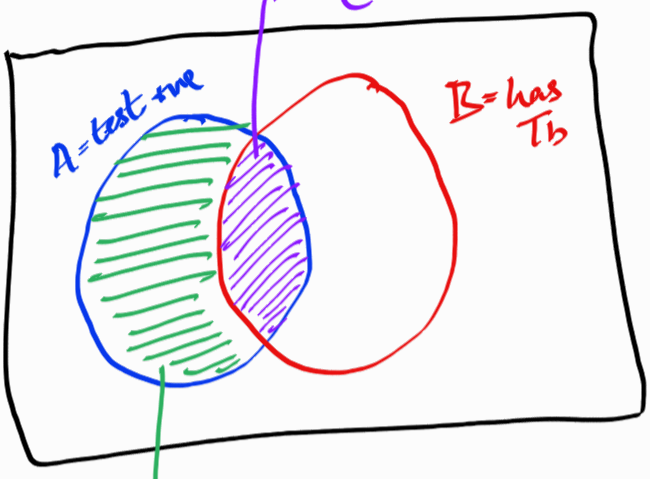
Via Bayes:

$$Pr(\text{has Tb} \mid \text{test +ve}) = \frac{Pr(\text{test +ve} \mid \text{has Tb}) \cdot Pr(\text{has Tb})}{Pr(\text{test +ve})}$$

$Pr(\text{test is accurate}) = 0.95$; $Pr(\text{has Tb}) = 0.0005$; $Pr(\text{test +ve}) = ?$

positive → +ve

negative → -ve



$Pr(\text{test +ve} \mid \text{has Tb}) = Pr(\text{test is accurate})$

$Pr(\text{test +ve} \mid \text{no Tb}) = Pr(\text{test inaccurate})$

$Pr(+ve) = Pr(\text{test is accurate}) + Pr(\text{test is inaccurate})$

$$Pr(+ve) = Pr(Tb)\,Pr(+ve \mid Tb) + Pr(\overline{Tb})\,Pr(+ve \mid \overline{Tb})$$

have our denominator in Bayes

$$Pr(\text{has Tb} \mid \text{test +ve}) = \frac{Pr(\text{test +ve} \mid \text{has Tb}) \cdot Pr(\text{has Tb})}{Pr(\text{test +ve})}$$

$$= \frac{0.95 \times 0.0005}{Pr(Tb)\,Pr(+ve \mid Tb) + Pr(\overline{Tb}) \cdot Pr(+ve \mid \overline{Tb})}$$

$$= \frac{0.95 \times 0.0005}{0.0005 \times 0.95 + \underbrace{0.9995}_{(1 - Pr(Tb))} \times 0.05}$$

0.0005 ↓ has Tb; 0.95 ↓ test accurate; 0.05 → $Pr(\text{test inaccurate})$

$$Pr(\text{has Tb} \mid \text{test +ve}) \approx 0.0094$$

$$\approx 0.94\%$$

↳ not very likely

③ Supplier A : 3000 products, 4% defective

Supplier B : 2400 products, 7% defective

Goal: find $Pr(B \mid \text{defective}) = Pr(\text{product came from supplier B} \mid \text{product was defective})$

Soln: $$Pr(B \mid def) = \frac{Pr(def \mid B) \cdot Pr(B)}{Pr(def)}$$

$$= \frac{0.07 \times \frac{2400}{5400}}{Pr(A) \cdot Pr(def \mid A) + Pr(B) \cdot Pr(def \mid B)}$$

$$= \frac{0.07 \times 24/54}{\frac{3000}{5400} \cdot 0.04 + \frac{2400}{5400} \cdot 0.07}$$

Expectation:

eg: fair coin toss → H means you get \$1

T means you lose \$2

$$\frac{1}{2}(\$1) + \frac{1}{2}(-\$2) = -\frac{\$1}{2}$$ → what you expect to gain playing this game. ⇒ game is a loss.

weighted average

Avg: $\frac{1 + (-2)}{2} = \frac{-1}{2}$

Formally

X is a random variable
↓
depends on chance

$X \to 1$ w/ prob $1/2$
$X \to -2$ w/ prob $1/2$

Expectation is "average / expected" value of the random variable
↓ ↓
imagine running the experiment many many times. number

$$E[X] = \underbrace{Pr(s_1) \cdot X_1 + Pr(s_2) \cdot X_2 + \cdots}_{\text{each possible outcome shows up once.}}$$

$$= \sum_{s_i \in S} P(s_i) \cdot X_i$$

eg: Expected roll of 6-sided dice. → $X_i = 1, 2, 3, 4, 5, 6$

$$E[\text{roll}] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \cdots + \frac{1}{6} \cdot 6$$

($\frac{1}{6}$ → $P(\text{roll}=6)$; 6 → value if roll = 6)

$Pr(\text{roll}=1)$

$$= \frac{1+2+\cdots+6}{6} = \frac{21}{6} = \boxed{3.5}$$

Eg New game: 3 sided die → A, B, C written on it.

A → you win $1

B → toss coin → H → $0.5
↘ T → −$0.6

C → you lose $0.9

$$E[\text{gain}] = P_r(A)\cdot X_A + P(B)\cdot X_B + P(C)\cdot X_C \rightsquigarrow E[Y]$$

$$E[\text{gain}] = \frac{1}{3}\cdot 1 + \frac{1}{3}\left[\frac{1}{2}\cdot 0.5 + \frac{1}{2}(-0.6)\right] + \frac{1}{3}(-0.9)$$

$$= \frac{1}{3}\left[1 + \frac{1}{4} - \frac{3}{10} - \frac{9}{10}\right]$$

$$= \frac{1}{3}\left\{\frac{10 + 2.5 - 3 - 9}{10}\right\}$$

$$= \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{10} \rightarrow +ve$$

. You could play this game.

Eg. Coin tosses $P_r(H) = 2/3$, $X_{Head} = 1$, $X_{Tail} = -1$

Expected value of tossing 3 coins → value of each toss is added.

S_i	$P_r(s_i)$	X_i
TTT	$(1/3)^3$	−3
TTH	$(1/3)^2\,2/3$	−1
THT	$(1/3)^2\,2/3$	−1
THH	$(2/3)^2\,1/3$	1
HTT	$(1/3)^2\,2/3$	−1
HTH	$(2/3)^2\,1/3$	1
HHT	$(2/3)^2\,1/3$	1
HHH	$(2/3)^3$	3

$$E[3\text{ coins}] = \frac{1}{27}(-3) + \frac{2}{27}(-1) + \frac{2}{27}(-1) + \frac{4}{27}(1) + \frac{2}{27}(-1) + \frac{4}{27}\cdot 1 + \frac{4}{27}\cdot 1 + \frac{8}{27}\cdot 3$$

$$= \frac{7\cdot 3}{27} + \frac{6\cdot 1}{27} = \frac{27}{27} = \boxed{1}$$