Oct 29

Admin

- PS6 → Oct 31, Sat
- Project Proposal → Nov 1, Sun
- Exam 2 → Nov 5, Thu
  2's comp, Sets, Counting, Prob (until today)
- Recitation this week: Prob practice + Project help
- Practice Exam 2: Friday evening
- Monday lecture: Prob wrapup, Exam review

Agenda

- Review prob
- Uniform distribution
- Multiple events
  - ↳ Joint prob
  - ↳ Independence
  - ↳ Conditional prob

Probability

Experiments: try things, (lots of) well-defined outcomes

Sample Space = S

Events: a set of specific outcomes of interest.

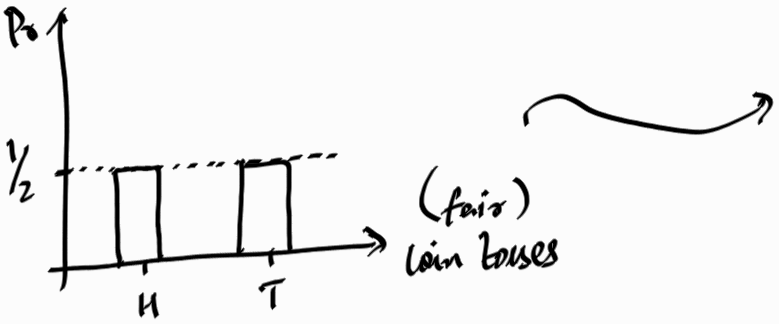
Prob of Event E

$$Pr(E) = \frac{|E|}{|S|}$$

Prob laws via set/counting

- $Pr(\bar{E}) = 1 - Pr(E)$
- $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$
- $Pr(A) + Pr(\bar{A}) = 1$

Sum of probs of all possible outcomes is always 1.



Uniform distribution

- if all outcomes are equally likely → uniform dist.
- if n possible outcomes exist each has prob $= \frac{1}{n}$

eg. Fair coin tossing
Uniform w/ $n=2$

eg. Fair 6-sided die
Uniform w/ $n=6$

eg. Pick a card from shuffled 52-card deck.
Uniform w/ $n=52$
$\Rightarrow Pr(\text{A of spades}) = 1/52$

Multiple Events

① Roll 20 sided die & toss a fair coin

Que: $Pr(\underbrace{\text{coin is H}}_{A} \;\&\; \underbrace{\text{roll is 1 or 20}}_{B}) = Pr(A \cap B)$

Both happening → (H,1) or (H,20)

other outcomes: (T, _) or (H, x) where $x \neq 1$ or 20

$$Pr(A \cap B) = \frac{|\{(H,1),(H,20)\}|}{40} = \frac{2}{40} = \boxed{\frac{1}{20}}$$

40 → all possible outcomes → $|\{H,T\} \times \{1,2,\ldots,20\}|$

Alternate

$Pr(A \cap B) = Pr(A) \cdot Pr(B)$

$= \frac{1}{2} \cdot \frac{2}{20}$

$= \boxed{\frac{1}{20}}$

(2) A bin has 10 balls: 7 are white, 3 red balls

"Experiment": pick balls from the bin one after other (without replacement)

Que. $Pr(\underbrace{1^{st} \text{ ball} = W}_{A} \text{ and } \underbrace{2^{nd} \text{ ball} = W}_{B}) = Pr(A \cap B)$

if product rule

$Pr(A \cap B) = Pr(A) \cdot Pr(B) = \frac{7}{10} \cdot \frac{7}{10}$

$Pr(A)$: ball is white; $Pr(B)$: ball is white

→ not quite right. - 1 white ball already out

more correct

$Pr(A \cap B) = \frac{7}{10} \cdot \frac{6}{9}$

$Pr(2^{nd} \text{ ball } W \text{ if we know } 1^{st} \text{ ball is } W)$ → 2^{nd} ball prob DEPENDS on 1^{st} ball

1^{st} ball	(W, R)	2^{nd} ball white
W →	(6, 3) remain	→ $\frac{6}{9}$ prob
R →	(10, R) (7, 2) remain	→ $\frac{7}{9}$ prob

formally

$Pr(2^{nd} \text{ ball } W \mid 1^{st} \text{ ball is } W)$ → Conditional probability

"given"

$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$

$Pr(A \cap B) = Pr(B) \cdot Pr(A|B)$

$= Pr(1^{st} \text{ ball is } W) \cdot Pr(2^{nd} \text{ ball is } W \mid 1^{st} \text{ ball is } W)$

$= \frac{7}{10} \cdot \frac{6}{9}$

Direct Approach

$$Pr(1^{st}=W \cap 2^{nd}=W) = Pr((W,W)) = \frac{|\{(x,y) \mid x=W, y=W\}|}{|\{(x,y)\}|}$$

$$= \frac{7 \cdot 6}{10 \cdot 9}$$

- Direct Approach always feasible
- If multiple events : figure out (in) dependance
  - if A & B independant ⇔ if & only if

$$Pr(A \cap B) = Pr(A) \cdot Pr(B) \quad \text{[Product rule]}$$

  - if A & B are not independant → No Product rule
    ↳ usually conditional prob.

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

---

4 2 bins → A → 7 W, 3 R
B → 4 W, 6 R

$$Pr(A=W \ \& \ B=W) \underset{\downarrow}{=} Pr(A=W) \cdot Pr(B=W) = \frac{7}{10} \cdot \frac{4}{10}$$

Independance!

bins separate : can't influence each other

Joint Probability Table of A & B (same as above)

	A = Red	A = White	$Pr(B=_)$ ↓
B = Red	$\frac{3}{10} \cdot \frac{6}{10}$	$\frac{7}{10} \cdot \frac{6}{10}$	$\frac{6}{10}$
B = White	$\frac{3}{10} \cdot \frac{4}{10}$	$Pr(A=W \cap B=W) = \frac{7}{10} \cdot \frac{4}{10}$	$\frac{4}{10}$
$Pr(A=_)$ →	$\frac{3}{10}$	$\frac{7}{10}$	

2nd look at Conditional Prob

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} \qquad Pr(B|A) = \frac{Pr(B \cap A)}{Pr(A)}$$

$$Pr(A|B) \cdot Pr(B) = Pr(A \cap B) = Pr(B \cap A) = Pr(B|A) \cdot Pr(A)$$

Bayes Rule / Bayes Theorem

$$Pr(A|B) \cdot Pr(B) = Pr(B|A) \cdot Pr(A) \Rightarrow Pr(A|B) = \frac{Pr(B|A) \cdot Pr(A)}{Pr(B)}$$

Examples

① 1 bin, 6 W balls, 4 R balls

Que What is prob of ≥ 2 W balls out of 3 total if we know 1st ball was W?

Dependant events

$$P_r(\geq 2W \mid 1^{st}=W) = \frac{P_r(\geq 2W \cap 1^{st}=W)}{P_r(1^{st}=W)}$$

$P_r(1^{st}=W) = \frac{6}{10}$

WW_ OR WRW

$$= \frac{\frac{6}{10}\cdot\frac{5}{9} + \frac{6}{10}\cdot\frac{4}{9}\cdot\frac{5}{8}}{\frac{6}{10}} = \frac{5}{9} + \frac{4}{9}\cdot\frac{5}{8}$$

$$= \frac{5}{9}\cdot\frac{12}{8} = \boxed{\frac{5}{6}}$$

② 3 bins: A → 7W, 3R
B → 6W, 4R
C → 5W, 5R

Que Prob that if 1 ball picked from each bin, all are the same color?

$$P(WWW \text{ OR } RRR) = P_r(WWW) + P_r(RRR)$$

$$= \frac{7}{10}\cdot\frac{6}{10}\cdot\frac{5}{10} + \frac{3}{10}\cdot\frac{4}{10}\cdot\frac{5}{10}$$

$$= \frac{7}{10}\cdot\frac{3}{10} + \frac{3}{10}\cdot\frac{2}{10} = \boxed{\frac{27}{100}}$$

③ Rolling 2 dice, 6-sided; **Que** What is prob that the sum of dice rolls is 7 or 8?

7 → 1+6, 2+5, 3+4 } 3
8 → 2+6, 3+5, 4+4 } 3

each is 2x since it could be 1+6 or 6+1 and so on for other pairs

$$\frac{3\cdot 2}{36} + \frac{3\cdot 2}{36} = \boxed{\frac{1}{3}}$$

$\frac{3\cdot 2}{36}$: sum of 7; $\frac{3\cdot 2}{36}$: sum of 8