Oct 28

## Admin

- PS6 → Oct 31, Sat
- Project Proposal → Nov 1, Sun
- Exam 2 → Nov 4, Wed
  Is comp, Sets, Counting, Prob (until today)
- Recitation this week: Prob practice + Project help
- Practice Exam 2: Friday evening.
- Monday lecture → Prob wrapup
  Exam review

## Agenda

- Review of prob
- Uniform distribution
- Multiple events
  - ↳ Joint prob.
  - ↳ Independence
  - ↳ Conditional prob.

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## Probability

Experiments: try things, (lots of) well-defined outcomes
- ↳ Sample space = $S$

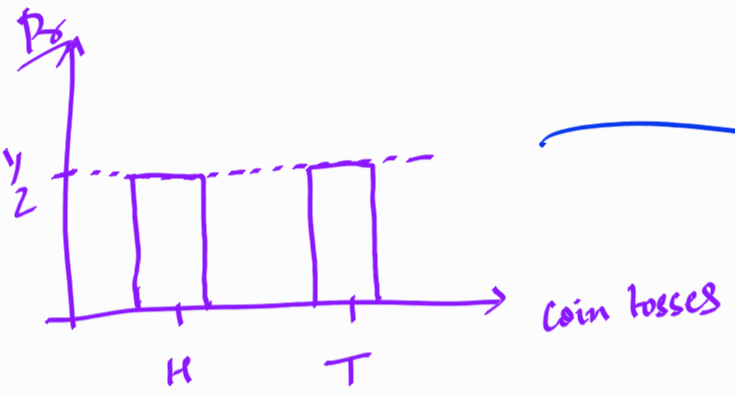
Event: a set of specific outcomes of interest.

Prob laws via set or counting laws

Probability of an event $E$

$$Pr(E) = \frac{|E|}{|S|}$$

- $\rightarrow Pr(\bar{E}) = 1 - Pr(E)$
- $\rightarrow Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$
- $\rightarrow Pr(A) + Pr(\bar{A}) = 1$
  - ↳ sum of probs. $= 1$




## Uniform distribution

- if all outcomes are equally likely, then it is a uniform distribution
- if n diff. possible outcomes

$Pr(\text{any 1 outcome}) = \frac{1}{n}$

eg: Fair Coin tossing
Uniform w/ 2 possible outcomes

eg. fair 6-sided dice
Uniform w/ 6 outcomes

eg. Pick a card from the standard 52-card deck.
Uniform $\Rightarrow Pr(\text{any specific card}) = \frac{1}{52}$

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## Examples of Multiple things

① Roll 20 sided dice & toss a fair coin

Que: $Pr(\text{coin} = H \text{ and dice roll} \in \{1, 20\})$

$= Pr(H \cap \text{roll} \in \{1, 20\}) = Pr(A \cap B)$

A: $H$, B: roll $\in \{1, 20\}$

ways to get what we want: $(H, 1)$ or $(H, 20)$

other outcomes = $(T, -)$, $(H, x)$ where $x \neq 1$ or 20.

$$Pr(A \cap B) = \frac{|\{(H,1), (H,20)\}|}{|\text{all possible outcomes}|} = \frac{2}{2 \times 20} = \boxed{\frac{1}{20}}$$

all possible outcomes → Cartesian product of $\{H,T\} \times \{1, 2, \ldots, 20\}$

thought: does product rule just work.

↳ $Pr(A \cap B) = Pr(A) \cdot Pr(B)$

$\frac{1}{20} = \frac{1}{2} \cdot \frac{2}{20} = \frac{1}{20}$

↳ Yes (in this case)

eg. Consider a bin with 10 balls. 7 are blue, 3 are red.

"Experiment" is picking balls one after the other. (without replacing it)

Qu. $Pr(1^{st}$ ball = Blue and $2^{nd}$ ball = Blue)

Product rule = $Pr(\text{ball is blue}) \cdot Pr(\text{ball is blue})$

$= \frac{7}{10} \cdot \frac{7}{10}$

↳ this only works if balls were put back in.

More correct

$= Pr(1^{st} \text{ ball is blue}) \cdot Pr(2^{nd} \text{ ball is blue})$ ↘ not equal

$= \frac{7}{10} \cdot \left(\frac{6}{9}\right) \rightarrow \neq$ ↓ $Pr(2^{nd} = \text{blue} \mid 1^{st} = \text{blue})$

it is assuming 1st ball already blue.

$1^{st}$ ball

B → (B,R) (6,3) remaining → $\frac{6}{9}$ Prob

R → (B,R) (7,2) remaining → $\frac{7}{9}$ prob

} $2^{nd}$ ball outcome DEPENDS on first ball outcome

## Actual approach

$Pr(1^{st} \text{ ball} = \text{Blue} \text{ and } 2^{nd} \text{ ball} = \text{Blue})$

No Product rule because event are dependent

$$Pr((B,B)) = \frac{7 \cdot 6}{10 \cdot 9}$$

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$Pr(A \cap B) = Pr(A) \cdot Pr(B)$ iff A and B are independent event.

↳ Not true for dependent events.

## Conditioned events

$$Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)}$$

"prob of A given B

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Ex. 2 bins → A → 7 blue, 3 red
B → 5 blue, 5 red

$Pr(A=\text{blue} \cap B=\text{blue}) = Pr(A=\text{blue}) \cdot Pr(B=\text{blue})$

↓ via independence

because bins are separate
don't influence each other.

# Joint Probabilities of A & B

| | A = Red | A = blue | $Pr(B = ?)$ |
|---|---|---|---|
| B = red | $\frac{3}{10} \cdot \frac{5}{10}$ | $\frac{7}{10} \cdot \frac{5}{10}$ | $\frac{10 \cdot 5}{10 \cdot 10} = \frac{5}{10}$ |
| B = blue | $\frac{3}{10} \cdot \frac{5}{10}$ | $Pr(A = \text{blue} \cap B = \text{blue}) = \frac{7}{10} \cdot \frac{5}{10}$ | $5/10$ |
| $P(A = ?)$ | $3/10$ | $7/10$ | |

Marginal probabilities

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## Lookback at Conditional Prob.

$$Pr(A|B) = \frac{Pr(A \cap B)}{P(B)}$$

$$Pr(B|A) = \frac{Pr(B \cap A)}{P(A)}$$

$$Pr(A|B) \cdot Pr(B) = Pr(A \cap B) = Pr(B \cap A) = Pr(B|A) \cdot Pr(A)$$

## Bayes Rule / Bayes Theorem

$$P(A|B) \cdot Pr(B) = Pr(B|A) \cdot Pr(A) \Rightarrow Pr(A|B) = \frac{Pr(B|A) \cdot Pr(A)}{Pr(B)}$$

# Examples

① 1 bin, 6 blue balls, 4 red balls

Que: what is prob of at least 2 blue balls out of 3 if the 1st ball was blue?

$$Pr(\geq 2 \text{ blue} \mid 1^{st} \text{ blue}) = \frac{Pr(\geq 2 \text{ blue} \cap 1^{st} \text{ blue})}{Pr(1^{st} \text{ blue})} \quad (Pr(1^{st} \text{ blue}) = \tfrac{6}{10})$$

$\geq 2$ blue $\cap$ 1st blue: BB — or BRB

$$= \frac{\frac{6}{10}\cdot\frac{5}{9} + \frac{6}{10}\cdot\frac{4}{9}\cdot\frac{5}{8}}{6/10} = \boxed{\frac{5}{9} + \frac{4}{9}\cdot\frac{5}{8}}$$

Conditional prob.

Dependant events

② 3 bins: A = 7 Blue, 3 Red
B = 6 Blue, 4 R.
C = 5 Blue, 5 R.

Que: Prob of ball from each bin being all the same color (so all 3 Blue or all 3 Red)

$$P(BBB \text{ or } RRR) = \frac{7}{10}\cdot\frac{6}{10}\cdot\frac{5}{10} + \frac{3}{10}\cdot\frac{4}{5}\cdot\frac{5}{10}$$

Independant events: picking from different bins

③ Rolling 2 dice, 6-sided; Que: What is Prob that the sum of rolls is 8 or 9.

8 → 2+6, 3+5, 4+4

9 → 3+6, 4+5

each is ×2 since it can be 2+6 or 6+2 & so on.

$$\frac{3\cdot 2}{36} + \frac{2\cdot 2}{36} = \boxed{\frac{10}{36}}$$

$\frac{3\cdot 2}{36}$: sums to 8

$\frac{2\cdot 2}{36}$: sums to 9