Oct 19

Admin

- Project Proposal in ~2 weeks
- Exam 2 in ~2 weeks
- Tutoring

Agenda

- more counting.
- recap
- Combinations (& Permutations)
- Sum Rule

Recap

Product rule → Cartesian Product's cardinality
↳ count # of "ordered lists"

$\underbrace{x}_{1^{st}} \cdot \underbrace{x}_{2^{nd}} \cdot \underbrace{x}_{3^{rd}} \cdots \underbrace{x}_{n^{th}}$ → are allowed to repeat

$x \cdot x \cdots x = x^n$

$\underbrace{x}_{} \cdot \underbrace{(x-1)}_{} \cdot (x-2) \cdots \underbrace{(x-n+1)}_{n^{th}}$ → no repetitions

n things in n slots → $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$

"Division rule" → rearrange "AKSHAR" → $\frac{6!}{2!}$
↳ 2 copies of A, internal rearrangement shouldn't be counted.

Example: # of rearrangements of "MISSISSIPPI"

very diff

$8!$ vs. $4! \cdot 4!$

$\$75\frac{2}{3}\$27$ $\quad 4 \cdot 3 \cdot 2 \cdot 1$

$= 70$ vs 1

$$\frac{11!}{4! \cdot 4! \cdot 2!}$$

- $4!$ → to handle 4 copies of I
- $4!$ → deal with duplicates of letters S
- $2!$ → for duplicates of P.

Combinations: # of ways to pick k things out of n total (distinct) things

eg. 0–9 (10 digits)

pick 3.

$$\frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1}$$

$$\frac{10 \cdot 9 \cdot 8}{3!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{3! \cdot 7!} = \frac{10!}{3!\,7!}$$

In general

Pick k out of n

$$\frac{n!}{k!(n-k)!} = \binom{n}{k} = {}^nC_k = C(n,k)$$

"n choose k"

Permutations of k things out of n

$P(n,k) = {}^nP_k \rightarrow$ # of ways of rearranging k things from a set of n things

$$\underset{1^{st}}{n} \quad \underset{2^{nd}}{(n-1)} \quad (n-2) \quad \cdots \quad \underset{k^{th}}{(n-k+1)} = {}^nC_k \cdot k! = P(n,k)$$

Example How many 6-digit (integers) exist s.t. no 2 consecutive digits in the integer are the same

(the)

$9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 9^6$

131313 ✓

133113 ✗

0 is not valid choice

10 9 10 9 10 9

should be 9.

Eg. # of rearrangements of "ABCDE" s.t. (AE: vowels)

a) vowels always at either end.

2 3 2 1 1 → $2 \cdot 3! = 12$

$3!$

b) vowels never at the end.

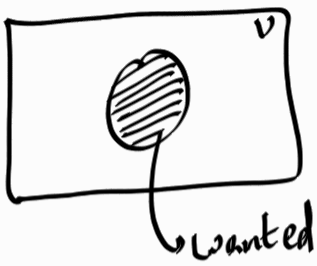
3 $3!$ 2 $= (3!)^2 = 36$

c) both ends can't be vowels.

all possible rearrangements → $5!$

$5! - 2 \cdot 3! = 120 - 12 = 108$

wanted = total − unwanted



$$|\text{wanted}| = |U| - |\overline{\text{wanted}}|$$

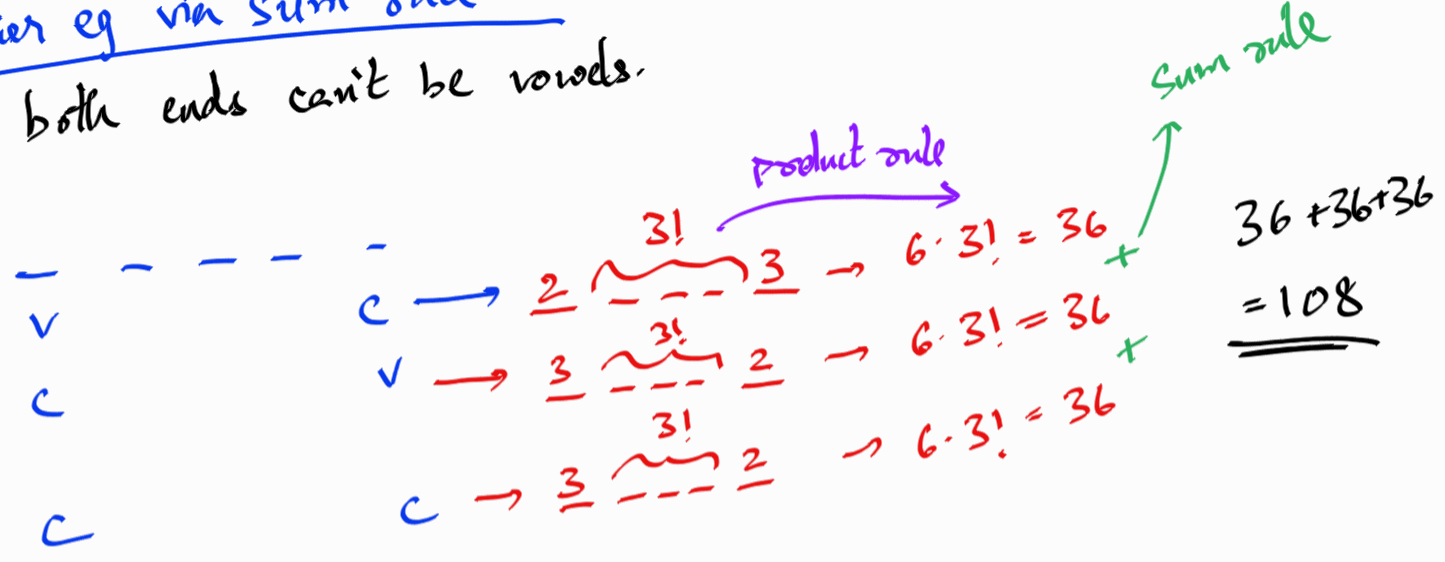
Sum Rule.

if A or B or C or ... can happen, but not together.
then just add # of ways each can happen.

$$|A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n| \quad \left[\text{if } \forall i,j \; |A_i \cap A_j| = 0\right]$$

earlier eg via sum rule

c) both ends can't be vowels.



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eg. # of 8 to 10 digit PINs?

$$10^8 + 10^9 + 10^{10}$$

↓ length 8 ↓ length 9 ↓ length 10

Example: MC Questions : 4 options, 100 questions

a) # of ways to answer all questions?

$$4^{100}$$

b) # of ways to answer some of the questions? → all blank is allowed.

Approach 1

$$4^0\binom{100}{0} + 4^1\binom{100}{1} + 4^2\binom{100}{2} + 4^3\binom{100}{3} + \cdots + 4^{99}\binom{100}{99} + 4^{100}\binom{100}{100}$$

$\binom{100}{0} = 1$, $\binom{100}{100} = 1$

5^{100} → Approach 2 (5 - pick A, B, C, D or leave blank)

c) # of ways to answer the questions s.t. 25 are A, 25 are B, 25 are C, 25 are D → options for each que.

X_4^{25} → flip direction of what we want.

$X 25^4$ ↗

$$\binom{100}{25}\binom{75}{25}\binom{50}{25}\binom{25}{25} = \frac{100!}{25!\,75!}\cdot\frac{75!}{25!\,50!}\cdot\frac{50!}{25!\,25!}$$

OR

$$\frac{100!}{25!\cdot 25!\cdot 25!\cdot 25!}$$

equal