Oct 19

Admin

- Tutoring
- Project proposals in ~2 weeks
- Exam 2 in ~2 weeks

Agenda

- More Counting
- Combinations (& Permutations)
- Sum Rule
- Examples

Recap

Product Rule → Cartesian Product

↳ "ordered list" is counted.

↳ $|A_1 \times A_2 \times \dots \times A_n| = |A_1| \times |A_2| \times \dots \times |A_n|$

$\frac{?}{1^{st}} \cdot \frac{?}{2^{nd}} \cdot \frac{?}{} \cdot \dots \frac{?}{n^{th}}$

if you can reuse x^n

if you can't reuse

$\frac{x}{1^{st}} \quad \frac{x-1}{2^{nd}} \quad \dots \quad \frac{x-(n-1)}{n^{th}}$

$x \cdot (x-1) \cdot (x-2) \dots (x-n+1)$

Division rule -

eg. rearrange "AKSHAR" → $\frac{6!}{2!}$

↳ A is duplicated twice
internal rearrangement of the As doesn't matter

eg. Count number of ways to rearrange: MISSISSIPPI

$$\frac{11!}{4!\,4!\,2!}$$

- $4!$ → handle duplicate i's
- $4!$ → duplicate S's
- $2!$ → duplicate P's

very diff. vs.

$4! \cdot 4!$ $\quad$ $8!$

$4 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ $\quad$ $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

2

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Combinations: Pick 3 digits (distinct) from $0, 1, \ldots, 9$

$$\underline{10} \cdot \underline{9} \cdot \underline{8} \;\Big|\; \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$$

← all possible rearrangements

$10 \cdot 9 \cdot 8$ → pick 1st three.

$\frac{10 \cdot 9 \cdot 8}{3!}$ → internal ordering doesn't matter. $= \frac{10!}{7! \cdot 3!}$

In general: k sized subset of n distinct things can be made in $\frac{n!}{k!\,(n-k)!}$ ways

$= \binom{n}{k} = {}^nC_k$

"n choose k"

Permutations of k things out of n

$\underbrace{\underline{n} \; \underline{n-1} \; \underline{n-2} \; _ \; _}_{k \text{ slots}} = n \cdot (n-1) \cdots (n-k+1) = \binom{n}{k} \cdot k! = P(n,k) = {}^nP_k$

q. 26 letters → make up 8 letter words. (w/o repeating letters)

↳ # of ways: $P(26, 8)$

$= {}_{26}C_8 \cdot 8!$

↳ make up 8 letter words with exactly 3 of them (vowels) distinct.

↳ ① choose 3 vowels → ${}_5C_3$

② -- 5 consonants → ${}_{21}C_5$

③ rearrange all 8. → $8!$

total: $\binom{5}{3}\binom{21}{5} \cdot 8!$

why not

— $5! \cdot 3!$ instead of $8!$ in step ③?

↓ ↓

rearranges vowels & cons. separately.

— ${}_5P_3 \cdot {}_{21}P_5$ for the full count? ↳ exactly same issue.

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q. How many "words" of at least length 8, at most length 10.

↓

any letters repetition allowed.

8-letter "words" ↳ $26 \cdot 26 \cdots = 26^8$ | 9-letter words ↳ 26^9 | 10-letter words ↳ 26^{10}

Sum Rule: any one of many distinct choices: add # of ways in each choice.

↳ # of 8↔10 letter words = $26^8 + 26^9 + 26^{10}$

Sum Rule is special case of P.I.E.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

if this is 0 we get sum rule.

In general

$$|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n| \quad \left(\text{if } |A_i \cap A_j| = 0 \ \forall i,j\right)$$

Examples.

① How many 5-digit integers s.t. no 2 consecutive digits are same.

$\underline{9}\ \underline{9}\ \underline{9}\ \underline{9}\ \underline{9}$

$\hookrightarrow 9^5$

① 1st digit $\neq 0$

② "next digit" $\neq$ "prev digit"

② MC Questions: 4 options each. 100 questions.

a). # of ways to answer all of these? $\rightarrow 4^{100}$

b) - - - - - - some of these? $\rightarrow$ leaving all blank allowed

$$4^{100}\binom{100}{0} + 4^{99}\binom{100}{1} + 4^{98}\binom{100}{2} + \dots + 4^2\binom{100}{98} + 4^1\binom{100}{99} + 4^0\binom{100}{100}$$

$\binom{100}{1}$: # of ways to leave 1 blank

$$= 5^{100} \quad \left(\text{pick } A, B, C, D \text{ or leave blank}\right)$$

(3) # of ways to answer 100 ques. s.t. 25 are answered as A.

25 ____ B

25 _ _ C

25 _ _ _ D

pick 25 out of 100 → A
$\binom{100}{25}$

pick 25 out of remaining → B
$\binom{75}{25}$

_ _ _ _ → C
$\binom{50}{25}$

pick 25 out _ _ _ → D
$\binom{25}{25} = 1$

Do all ⇒ product rule

$$\binom{100}{25}\cdot\binom{75}{25}\cdot\binom{50}{25}\cdot\binom{25}{25}$$

(4) No 2 consecutive questions have same answer.

$$\underline{4}\cdot\underline{3}\cdot\underline{3}\cdots = 4\cdot 3^{99}$$

Eg. rearrange "A B C D E" s.t. no vowels at either end.

$$\underline{3}\ \overset{3!}{\frown}\ \underline{2} = 3\cdot 3\cdot 2\cdot 1\cdot 2 = 36$$

$\binom{3}{2}\cdot 2!\cdot 3!$

↳ which consonants at either end.

Alternate: Total rearrangements − rearrangements w/ ≥1 vowel at an end point.

$5! - (2\cdot 3!\cdot 1 + 2\cdot 3!\cdot 3 + 3\cdot 3!\cdot 2)$

$120 - (12 + 36 + 36) = 120 - 84 = \underline{36}$