CS1800 Discrete Mathematics

Lecture Note 1: Propositions, Implications, Predicates, Quantifiers

Strange & Varma *Date: September 2026*

Note: Use this as supplementary to the handwritten notes from the lectures.

1 Propositions

Definition 1.1 (Proposition)**.** *A statement that is either a True statement or a False statement.*

Remark. *Take careful note of implicit assumptions that you might be making when considering propositions. Especially, English (and all natural (human) languages) are poor at being precise in their meaning.*

1.1 AND Operations

Two propositions, represented by boolean variables a, b can be combined using the AND operator: $a \wedge b$ which is a boolean expression which is True only when both a and b are True. In all other cases, $a \wedge b$ is False.

a	b	$a \wedge b$
F	F	F
F	T	F
T	F	F
T	T	T

Table 1: AND operation Truth Table

1.2 OR Operations

Boolean variables a, b can also be combined using the OR operator: $a \vee b$ which is a boolean expression which is True when either of a or b is True. If both are False, then $a \wedge b$ is False.

a	b	$a \vee b$
F	F	F
F	T	T
T	F	T
T	T	T

Table 2: OR operation Truth Table

1.3 NOT Operations

A boolean variable's truth value can be negated, using the negation/NOT operation: $\neg a$ which is a boolean expression which flips the truth value of a. It is True when a is False and False when a is True.

a	$\neg a$
F	T
T	F

Table 3: NOT operation Truth Table

1.4 Laws of Boolean Algebra

There are a few laws that can be defined/derived for the operations above. They are given in Table 4.

Commutative Laws	$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$
Associative Laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive Laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity Laws	$p \wedge T \equiv p$ $p \vee F \equiv p$
Annihiltor/Domination Laws	$p \wedge F \equiv F$ $p \vee T \equiv T$
Idempotence Laws	$p \wedge p \equiv p$ $p \vee p \equiv p$
Complement/Negation Laws	$p \wedge \neg p \equiv F$ $p \vee \neg p \equiv T$
Double Negation Law	$\neg\neg p \equiv p$
Absorption Laws	$p \wedge (p \vee q) \equiv p$ $p \vee (p \wedge q) \equiv p$
De Morgan's Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Table 4: Laws of Boolean Algebra.

2 Implications: $a \implies b$

In $a \implies b$, the first proposition/variable is often called the hypothesis (a in this case) and the second is called the consequence (b in this case). The implication operation can be thought of in many ways, whichever makes best intuitive sense.

- If the hypothesis is true, then the consequence must be true.
- If/Whenever a happens, b always happens.
- You can think of these as two dominoes lined up next to each other. If you push the a domino and make it fall, then the b domino is forced to also fall. Here, we associate falling with the True value.
- You can think of it as a being True triggering b to also be True.

Whichever intuition is easiest to grasp should be kept at the forefront while the others should still make sense. In some situation one intuition is easier to use than another.

2.1 Truth Table for Implication

The truth table (Table 5) for implication allows us to see a few quirks/subtleties of the implication operation in action.

Table 5: The Truth Table for $p \implies q$

p	q	$p \implies q$
F	F	T
F	T	T
T	F	F
T	T	T

We can re-write the implication using the usual AND, OR, NOT operations via the standard DNF/CNF construction. Consider $\neg(p \implies q) \equiv (p \wedge \neg q)$ by looking at the single False row in the Table 5. This suggests, via De Morgan's law, that $p \implies q \equiv \neg p \vee q$.

In words, this says that either p doesn't happen or q happens which is in fact what the implication says. If p doesn't happen, then we do not really care what goes on with q. When thinking of the OR operation as "at least one T", we can see that once $\neg p$ is True, the OR operation can "end". We do not need to bother with q. However, in the case where $\neg p$ is False, which is to say that p is True, then we need q to be true. So whenever the first part of the OR operation results in False (the hypothesis is True), we are forced to look at the second part and have the consequence be True.

Original statement	$p \implies q$	Equivalent
Contrapositive	$\neg q \implies \neg p$	
Converse	$q \implies p$	Equivalent
Inverse	$\neg p \implies \neg q$	

Table 6: Possible transformations of implication statements.

2.2 Contrapositive

Given an implication $p \implies q$ its contrapositive is given by $\neg q \implies \neg p$. Using the truth tables, or using the equivalence $p \implies q \equiv \neg p \vee q$, it is straightforward to see that the contrapositive and the original statement must be equivalent.

$$\begin{aligned} \neg q \implies \neg p &\equiv \neg(\neg q) \vee \neg p \\ &\equiv q \vee \neg p \\ &\equiv p \vee \neg q \\ &\equiv p \implies q \end{aligned}$$

The contrapositive is useful in proofs. Sometimes it is easier to prove the contrapositive compared to proving the original statement.

2.3 Converse

Given an implication $p \implies q$ its converse is given by $q \implies p$. Usually, the converse and the original are not equivalent.

If both the original statement and the converse are true, then we have $(p \implies q) \wedge (q \implies p)$ for which we have the notation $p \iff q$.

2.4 Inverse

Given an implication $p \implies q$ its inverse is given by $\neg p \implies \neg q$. This is the contrapositive of the converse. Therefore, the converse and inverse are always equivalent in truth value.

3 Predicates

A predicate takes an input and outputs True if input has some property, False otherwise.

$$P(x) = \begin{cases} 1 & x \text{ has property } P \\ 0 & otherwise \end{cases}$$

Examples of (input, properties) pairs:

- (Integers, Is the number even?) $\longrightarrow Even(n)$
- (Integers, Is the number odd?) $\longrightarrow Odd(n)$
- (Natural numbers, Is the number prime?) $\longrightarrow Prime(n)$
- (Natural numbers, Is the number composite?) $\longrightarrow Composite(n)$
- (Shapes, Is the shape a square?) $\longrightarrow Square(n)$
- (Shapes, Is the shape a Circle?) $\longrightarrow Circle(n)$
- (Shapes, Is the shape a Triangle?) $\longrightarrow Triangle(n)$
- (Pair of integers (d, n), Does d divide n without remainder?) $\longrightarrow Divides(d, n)$

A "Left-even" integer is an integer whose leftmost digit is even [Predicate $L(n)$]. A number n is even if its rightmost digit is even [Predicate $E(n)$]. A number n is divisible by 5 if the rightmost digit is 0 or 5 [Predicate $F(n)$]. A "decent" integer is an integer which is "L-Even" *and* is either even or divisible by 5. Let $D(n)$ be the predicate for "decent" numbers. Write a formula $D(n)$ in terms of $L(n), E(n), F(n)$.

$$D(n) = L(n) \wedge (E(n) \vee F(n))$$

We always want the number to be "Left-even", and at the same time, the number should have its rightmost digit even or equal to 0 or 5. So we have a AND between $L(n)$ and the rest and a OR between $E(n)$ and $F(n)$ since either of these being true is sufficient.

Given n numbers $a_1, a_2, \ldots, a_n$, write a formula p which is true if all n numbers are "decent":

$$\begin{aligned} p &= D(a_1) \wedge D(a_2) \wedge \cdots \wedge D(a_{n-1}) \wedge D(a_n) \\ &= L(a_1) \wedge \Big(E((a_1) \vee F(a_1)\Big) \wedge L(a_2) \wedge \Big(E((a_2) \vee F(a_2)\Big) \wedge \ldots \\ &\quad \wedge L(a_{n-1}) \wedge \Big(E((a_{n-1}) \vee F(a_{n-1})\Big) \wedge L(a_n) \wedge \Big(E((a_n) \vee F(a_n)\Big) \end{aligned}$$

Note: this is a conjunction of disjunctive clauses $\longrightarrow$ CNF where each $D(a_i)$ is 2 clauses by itself.

4 Quantifiers

Remark. *For this section in particular, it is very useful to think in terms of:*

- *AND means "ALL must be True".*
- *OR means "At least one/something must be True".*
- *NOT is "Flip/Swap/Opposite".*

Consider the scenario where I want to say: If k is an integer $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, then the integer $2k$ is even. We'll also use: $Z(k)$: predicate for k being an integer; $E(n)$: predicate for n is even. Now, what we want to say is simply: $Z(k) \implies E(2k)$ except for the fact that we want to say it for every possible integer value of k. A long-winded way to write it is:

$$\Big(Z(0) \implies E(2 \cdot (0)\Big) \wedge \Big(Z(1) \implies E(2 \cdot (1)\Big) \wedge \Big(Z(-1) \implies E(2 \cdot (-1)\Big) \tag{1}$$

$$\wedge \Big(Z(2) \implies E(2 \cdot (2)\Big) \wedge \Big(Z(-2) \implies E(2 \cdot (-2)\Big) \wedge \Big(Z(3) \implies E(2 \cdot (3)\Big) \wedge \ldots \tag{2}$$

This works as we want it to, but it is very long and looks a little ugly. However, the primary issue is that we will never finish writing it since there are an infinite number of integers. We want to say that something is true, "for all" *things*. Which is the same as saying that we want the AND of a lot of *things* to be true. The example above: "For all integers" k, the integer $2 \cdot k$ is even. Similarly, we may want to say that the OR of a lot of things is true.

To capture such notions, we use what are called quantifiers[1]. These are versions of the AND and OR operators which apply to a large collection of things instead of just two variables. Instead of saying $P(a_1) \wedge P(a_2) \wedge \ldots \wedge P(a_{n-1}) \wedge P(a_n)$ (where $P(\cdot)$ is some predicate), we can instead say *for all values* a_i with i ranging from 1 to n, $P(a_i)$. This rewriting/generalization/reparameterization of the symbols provide us with a lot more power than explicitly writing out each and every variable along

[1] *quantifiers:* from quantity, from quantus; how many

with the AND operation. In particular, it allows us to do this on collections that are infinite, like the integers from our earlier example.

Using quantifiers, we can say, "For all" k such that $Z(k) \implies E(2k)$. This also has the notation $\forall$, which is an inverted A (from **A**ll), allowing us to write $\forall k\colon Z(k) \implies E(2k)$. This succinct representation conveys the same meaning as Equation (1).

Just as the "for all" quantifier, we also have a "there exists" quantifier, which is used to convey that at least one thing is true allowing us to extend the OR operator to large collections, exactly how "for all" extends the AND operator to large collections. This allows writing $E(n) \implies$ "there exists" k such that $Z(k) \wedge (n = 2k)$. This statement can be read as: "If n is even, then there is some value of k such that k is an integer and n is twice the value of k". This quantifier has the notation $\exists$, which is an inverted E (from **E**xists), allowing us to write the even more succinct expression: $E(n) \implies \exists k\colon Z(k) \wedge (n = 2k)$.

4.1 Negating Quantifiers Or: How to Apply De Morgan's Law on Larger Scales

$$\neg\forall x : P(x) \iff \exists x : \neg P(x)$$

$P(x)$ is true for all x, is False $\iff$ There is some x for which $P(x)$ is false.

$$\neg\exists x : P(x) \iff \forall x : \neg P(x)$$

There is an x for which $P(x)$ is true, is False $\iff$ For all x, $P(x)$ is false.

If you look closely, you'll see that this is simply De Morgan's law applied to the quantifier versions of the AND and OR operators.

4.2 Playing Games Analogy for (Alternating) Quantifiers

One analogy for thinking about quantifiers is to think of playing a game and the quantifiers are restricting the moves that are allowed. This is an especially powerful way of looking at quantifiers of the form $\forall p \exists q$ such that r. This can be interpreted as a two player game where the "opponent" makes any move ($\forall p$) and in response to that we know that we will always have some move ($\exists q$) such that r holds. You win the game if you can in fact find a move in response to your opponents move which makes r True.

When the order is flipped and we have $\exists p \forall q$ such that r, then we can keep the same idea except we choose and make the first move ($\exists p$) and we are confident enough in our move that we know that irrespective of what the opponent picks ($\forall q$), r will always be True. Whenever quantifiers alternate between $\forall$ and $\exists$, this game analogy can be useful with our moves corresponding to the $\exists$ quantifiers and opponent moves corresponding to $\forall$ quantifiers. Depending on the order of the quantifiers, sometimes we make the first move, sometimes the opponent makes the first move.

The above handles all quantifier scenarios except when there is only one quantifier in use (no alternation of the form $\forall\exists$ or $\exists\forall$). If the only quantifier is $\exists$, then the earlier analogy works and we have the last move and we simply need to find some winning move. If there is a single $\forall$ quantifier, then it is as if we have already played our moves and are confident that whatever move the "opponent" plays, we will win.

4.3 Using Quantifiers

In the following, we will interpret logical expressions in English, translate English statements into logical expressions as well as attempt to determine the truth value of the logical expressions where possible. **Interpretation and translation to/from logical expression is more important than determining the Truth value.** We provide a basic truth value assessment to help cement the interpretation/translation; the "proof" of correctness is not the primary focus.[2]

Note: The domain for our quantifiers is integers when we use variables n, m and real numbers when we use variables x, y. Predicates $Even(n), Odd(n)$ are True when n is Even, Odd respectively.

1. $\forall n,\ n > 0$
   In English: For any *chosen integer n, the integer is positive.*
   This is false since if we look at *every* integer, then it will include negative integers, as well as zero for all of which $n \not> 0$

2. $\forall n, n^2 \geq n$
   In English: For any *integer n, the square of the integer is at least as large as the integer itself.*
   This is true. Since we look at n^2, we do not need to think of negative numbers separately since squaring them gives positive numbers which are always larger than the negative starting point. Zero and one are both true since in those cases $n^2 = n$. For positive numbers greater than 1, the squares are all larger. Thus, the statement is true for every integer value n.

3. $\forall x : x^2 \geq 0$
   In English: For any *real number x, the square of x is non-negative.*
   True. The square of 0 is 0, the square of positive numbers is positive and the square of negative numbers is also positive since it is the product of two negative numbers.

4. $\exists n,\ n + 37 = 73$
   In English: There is some *integer n, such that* $n + 37 = 73$
   We look through all integers searching for a n such that $n + 37 = 73$ which happens when $n = 73 - 37$. Thus, there is in fact a integer value which works and thus this is also a true statement.

5. $\exists n,\ n \cdot 73 = 37$
   In English: There is some *integer n such that* $n \cdot 73 = 37$
   If we consider all integers, then we will fail to find any value n such that $n \cdot 73 = 37$. The only solution to this equation is when $n = \frac{37}{73}$ which is not an integer. Thus, there is no integer which satisfies the requirement and hence there doesn't exist any such n.

6. $\forall x\ \exists y : x + y = 0$
   In English: For any *chosen real number x, there is always some value y such that* $x + y = 0$
   $\forall x$ means we should consider any arbitrary real number value x and the rest of the statement should hold true for any such value since $\forall$ is like the AND operator which is true for every possibility. Since it is true for every possibility, it is true for any arbitrary possibility. Once this arbitrary x has been chosen, the statement says that we must find a y value such that $x + y = 0$. This y value can be taken to be $-x$ which allows the remainder of the statement to be True.

7. $\exists x\ \forall y : x + y = 0$
   In English: There is some real number x such that for all real numbers y, the value of $x + y$ *is 0.*

[2] Although we are building up to that, in a few more lectures.

Here, the order of the "AND" and "OR" operations are flipped. We need to find some x value such that for any y value we may choose, $x + y = 0$. However, for every value y, there is a unique value $x = -y$ which satisfies $x + y = 0$. It is not possible to choose a value of x that is the same for all values of y and satisfy the equation $x + y = 0$ at the same time. Therefore, this statement is False.

8. $\exists x\ \forall y : xy = 0$
   In English: There is some real number x such that for all real numbers y, the value of xy is 0.
   The quantifiers are in the same order as the previous statement. However, in this case the equation which needs to be satisfied is $xy = 0$. This in fact allows us to choose a value of x which will work irrespective of the value of y (for all possible values). Setting $x = 0$ ensures that the product xy will be zero for all values of y. Thus this statement is True.

9. $\forall x\ \exists y : \frac{x}{y} = 1$
   In English: For any real number x there is some real number y such that $\frac{x}{y} = 1$.
   This statement is true since for an arbitrarily picked value of x, we can always set $y = x$ which would make the equation $\frac{x}{y} = 1$ to be True.

10. $\forall x\ \exists y : x - y = 1$
    This statement is True since for any arbitrarily chosen value of x, we can choose $y = x - 1$ such that $x - y = x - (x - 1) = 1$ satisfies the requirement.

11. $\forall n, m : Even(n + m) \iff [Even(n) \wedge Even(m)] \vee [Odd(n) \wedge Odd(m)]$
    In English: For any two integers n, m if their sum is even then one of two things must happen. First possibility is that both of n, m are even and the second possibility is that both of n, m are odd.
    This is a True statement which can be readily verified once you know that every even number can be written as $2k$ for some integer k and every odd number can be written as $2k + 1$ for some integer k.

12. $\forall n, m : Even(nm) \iff Even(n) \vee Even(m)$
    In English: For any two integers n, m the product nm is Even if and only if at least one of n and m is even.
    Also a true statement which can be readily seen once even numbers are written in the $2k$ form.

13. $\forall n, m : Odd(nm) \iff Odd(n) \wedge Odd(m)$
    In English: For any two integers n, m the product nm is odd if and only if both of n and m are odd.

14. i is an index between 1 and n such that a_i is an element in a list of integers.
    $Odd(a_1 + a_2 + \ldots a_{n-1} + a_n) \implies \exists i : Odd(a_i)$
    In English: If the sum of all integers $a_1 + a_2 + \ldots a_{n-1} + a_n$ is odd, then there must be some index i such that the integer a_i is odd.
    True. If all a_i were even, then the sum is $2k_1 + 2k_2 + \cdots + 2k_n = 2(k_1 + k_2 + \cdots + k_n)$ which is even. Thus, there has to be at least one odd integer a_i so that the overall sum is odd.

15. Every odd number n is the sum of an even number m_1 and an odd number m_2.
    In Logic: $\forall n \exists m_1, m_2 : (n = m_1 + m_2) \wedge Even(m_1) \wedge Odd(m_2)$
    True. Every odd number can be written as $2k + 1$ for some integer k. We can simply use $2k$ as m_1 and 1 as m_2.

16. An integer n is odd if and only if n^{2k+1} is odd for all integers k.
    In Logic: $Odd(n) \iff \forall k, Odd(n^{2k+1})$

True. Proof is beyond the scope of this lecture.[3]

17. For all integers n, n^{2k} is even for all integers k.
In Logic: $\forall n,\, k\colon Even(n^{2k})$
False. Proof is again out of scope of lecture.

18. $\forall n : \big[Odd(n^2) \iff Odd(n)\big] \wedge \big[Even(n^2) \iff Even(n)\big]$
In English: For any integer n, the square of n is odd if and only if n itself is odd and also n^2 is even if and only if n itself is even.
True.

19. $\forall i : Even(a_i) \implies Even(a_1 + a_2 + \ldots a_{n-1} + a_n)$
In English: For all values i, if a_i is even, then the sum of all a_i terms is also even.

20. Is the converse of the previous statement true? *The converse is:* $\forall i : Even(a_1 + a_2 + \ldots a_{n-1} + a_n) \implies Even(a_i)$
This is not True, since we could add an even number of odd numbers and the resulting sum would be even.

21. For any two numbers x, y, if $x^2 > y^2$ then either $x > y$ or $-x > -y$ but not both.
In Logic: $\forall x,\, y\colon x^2 > y^2 \implies (x > y) \oplus (-x > -y)$

22. For any two numbers x, y, $x^2 > y^2$ if and only if exactly one of $x > y$ and $-x > -y$ is True.
In Logic: $\forall x,\, y\colon x^2 > y^2 \iff (x > y) \oplus (-x > -y)$

23. Are the two previous statements equivalent?
No, the latter statement has a double implication while the former does not making the former a weaker statement since it does not mention the converse while the double implication does.

24. $\forall x, \exists y : x = y^2$
In English: For any real number x we can find a real number y such that x is the square of y.
False. Whenever x takes on negative values, there is no real number which on squaring will give the result x.

[3] For those interested, the result can be derived by utilizing statements above about $Odd(mn) \iff Odd(n) \wedge Odd(m)$.