CS1800 Discrete Structures

Solutions to Recitations 5

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- This recitation handout contains practice problems. You do *not* need to submit anything.
- We do recommend handwritting your solutions to completion. Do ensure that you are aware of how and why each step in the solution exists.
- For best effects, read through the questions before your recitation section and if possible attempt them on your own. This way you can compare your perspective to the TAs during the recitation.
- If you get stuck on a problem, use your recitation section to get help. Of course, you can also use office hours or post on Campuswire!
- The solutions to this recitation will be posted after Friday. Use those to verify your own attempts and for any clarification.
- If problems are marked $(*)$, those are fundamental and the others are additional practice using concepts from the fundamental practice problems in more complex ways.

Problem 1 (Counting cards)

There are 52 playing cards, 13 of each of the four sutis: Spades, Clubs, Hearts, and Diamonds. The 13 cards in each suit have values 2, 3, 4, ..., 9, 10, Jack (J), Queen (Q), King (K), and Ace (A). The last 4 (J, Q, K, A) are face cards while the others are numeric cards. The cards of Hearts and Diamonds are red in color while Spades and Clubs are black in color. So there are 26 red and 26 black cards.

When we draw cards, we draw them one by one. A hand of n-cards is a set of cards, without a specific ordering among them. Poker hands are a set of five (distinct) cards with certain properties.

(a) How many ways can you make a poker hand such that:

i. **(∗)** There are only red cards in the hand.

Solution:

26 red cards, pick 5:

$$\binom{26}{5}$$

■

ii. There are only even value (numeric) cards in the hand.

Solution:

Even cards: $\{2, 4, 6, 8, 10\}$ are 5. Across 4 suits, we get $5 \times 4 = 20$ such cards. Pick 5.

$$\binom{20}{5}$$

■

iii. There are exactly two face cards in the hand.

Solution:

Pick 2 face cards out of $|\{J, Q, K, A\}| \times |\{S, C, H, D\}| = 4 \times 4 = 16$ face cards. Then pick 3 more from remaining $52 - 16 = 36$ cards.

$$\binom{16}{2}\binom{36}{3}$$

■

iv. **(∗)** All cards in the hand are either cards with prime value or face cards.
So mixed hands are not allowed.

Solution:

Prime values: $\{2, 3, 5, 7\}$ across 4 suits are 16 cards. Pick 5 from these or 5 from the 16 face cards (via sum rule).

$$\binom{16}{5} + \binom{16}{5} = 2\binom{16}{5}$$

■

v. There is at least one card from each suit in the hand.

Solution:

Pick 1 from the 13 cards in a suit. Do this 4 times. Pick 1 out of the remaining 48 cards. Product rule to combine all of these.

$$\binom{13}{1}^4 \binom{48}{1} = 13^4 \cdot 48$$

The above is almost right. However, this counts certain things twice. The 5th card will be of the same suit as one of the first 4. This same hand will also be counted when the 5th card was instead chosen as part of the first 4 and the card of the same suit was instead chosen as the 5th card. So each possible hand is being counted twice. So we need to divide by two.

$$\binom{13}{1}^4 \binom{48}{1} \cdot \frac{1}{2} = 13^4 \cdot 24$$

■

vi. **(∗)** The hand is either a Four of a kind or a Royal Flush.
Four of a kind: there are 4 cards with the same value, one from each suit.
Royal Flush: values: 10, J, Q, K, A, all of the same suit.

Solution:

Four of a kind: Pick 1 out of 13 values, all 4 suits of it picked, then pick 1 from remaining 48 cards.

Royal Flush: Pick 1 out of 4 suits, values are forced.

Sum rule:

$$\binom{13}{1}\binom{48}{1} + \binom{4}{1} = 13 \cdot 48 + 4$$

■

vii. **(∗)** There are exactly two cards of the same value. *So other 3 are all distinct.*

Solution:

Pick 1 of 13 values, pick 2 of 4 suits for this value. Pick 3 more values from the remaining 12 (since exactly two cards of same value, so all others should be single cards). For each of these 3 values, pick their suit.

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3$$

■

viii. **(∗)** There are exactly two cards of the same value and exactly three of a different value.

Solution:

Pick 1 of 13 values for the pair, pick 2 out of 4 suits. Pick 1 out of 12 remaining values for triple, pick 3 out of 4 sutis.

$$\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{3}$$

■

ix. All cards are consecutive in value. According to the order: A, 2, 3, . . . , 9, 10, J, Q, K, A. *Yes, A shows up twice. Further, we are allowed to wrap around. So something like Q, K, A, 2, 3 is a valid ordering.*

Solution:

Once we pick the starting point of the consecutive cards, the rest are decided. Start can be any of the 13 cards since we can wrap around. Any of 4 suits for each of the 5.

$$\binom{13}{1} \cdot 4^5$$

■

x. All cards are consecutive in value. According to the order: A, 2, 3, . . . , 9, 10, J, Q, K, A. *Yes, A shows up twice. However, this time, we are NOT allowed to wrap around. So something like Q, K, A, 2, 3 is a NOT valid ordering.*

Solution:

Once we pick starting, cards are fixed. However, the "largest" starting can only be 10, since starting at J would result in wrapping around. So anything from A, 2, . . . , 10. That's 10 possibilities. Any of 4 suits for each of the 5.

$$\binom{10}{1} \cdot 4^5$$

■

(b) We draw 12 (distinct) cards in order. How many ways can we draw these cards such that:

i. **(∗)** Red and black cards alternate. It doesn't matter if red or black cards are at the start.

Solution:

It is either RBRBR.. or BRBRB.. In the first case, we need to permute 6 red cards and 6 black cards. Similarly for the second case. So 2 times permuting 6 red and 6 black cards.

$$P(26,6) \cdot P(26,6) \cdot 2$$

■

ii. 3 cards of each suit show up and all cards of the same suit are together.

Solution:

Order of suits 4!. Within cards of a suit, permute 3 out of 13 cards. Do this once for each suit.

$$4!\big(P(13,3)\big)^4$$

■

iii. **(∗)** 3 red cards show up together, and they are not on either end.
So the first and last cards are black.

Solution:

Permute 2 black cards for the ends. Permute 3 red cards, start them from one of 8 possible starting places (slots 2, 3, ..., 9). Permute 7 more black cards (from the 24 remaining black cards) for the remaining 7 slots.

$$P(26,2) \cdot P(26,3) \cdot 8 \cdot P(24,7)$$

■

iv. There are exactly 6 cards of Spades, with none being the first or last card drawn.

Solution:

Permute 2 of the 39 non-spade cards into the ends. Pick which 6 slots (out of 10 internal slots) are used by spades cards. Permute 6 of 13 spades into those slots. Then permute 4 from the remaining 37 non-spade cards into the 4 remaining slots.

$$P(39,2) \cdot \binom{10}{6} \cdot P(13,6) \cdot P(37,4)$$

■

v. **(∗)** All face cards show up after all numeric cards.

Solution:

The pattern can be: 0 Num/12 Face OR 1 Num/11 Face OR ... OR 11 Num/1 Face OR 12 Num/0 Face. There are 16 total face cards and hence 36 numeric cards. Permute the required number of numeric and required number of face cards. Slots don't need to be determined since numeric always show up first.

$$P(36,0) \cdot P(16,12) + P(36,1) \cdot P(16,11) + \cdots + P(36,11) \cdot P(16,1) + P(36,12) \cdot P(16,0)$$

Note that $0! = 1$ and $\binom{n}{0} = 1$. So $P(n, 0) = 1$. ■

vi. There are four consecutive aces (A) with their suits alternating red and black.

Solution:

Choose the starting point of the 4 aces from slots 1, 2, . . . , 8, 9. Suit order could be RBRB or BRBR, that's a factor of 2. Red internal order could be in 2 ways and black suit order could also be in 2 ways. So in total that $2 \cdot 2 \cdot 2$ ways for the 4 suits to be arranged so that red and black alternate. Once the Aces are placed, permute 8 cards from the remaining 48.

$$\binom{9}{1} \cdot 2 \cdot 2 \cdot 2 \cdot P(48, 8)$$

■

vii. **(∗)** The pattern of suits is not: SHCD SHCD SHCD.

Solution:

Permute three Spades into slots 1, 5, 9. Three Hearts into 2, 6, 10. Three Clubs into 3, 7, 11. Three Diamonds into 4, 8, 12.

$$\big(P(13, 3)\big)^4$$

■

Problem 2 (Balls in Bins)

There are a 100 identical balls in a bag. These must be placed into 4 bins named A, B, C, D. How many ways can these balls be placed into the bins given the following conditions:

(a) in total, if there are no restrictions, except all balls being placed in some bin.

Solution:

No restrictions, so standard stars and bars. Number of stars is number of balls = 100. Number of bars is 1 less than number of bins, so $4 - 1 = 3$. Bars are the dividers between bins, not the bins themselves. Arranging these 103 things is the same as choosing where the 3 bars show up out of 103 possible slots. Everything else will be the stars.

$$\binom{103}{3}$$

■

(b) **(∗)** if each bin must have at least one ball?

Solution:

Arrange all 100 stars. We need to place 3 bars. Between each pair of consecutive stars, we have one slot where a bar could possible be placed. That's a total of 99 possible slots[1] out of which we need to pick the 3 where the bars should go.

$$\binom{99}{3}$$

■

(c) **(∗)** if the combined total balls in bins A and B must be exactly equal to the combined total balls in bins C and D?

Solution:

If total balls in A and B is equal to that in C and D, then each has 50 balls. So the bar dividing B and C is placed immediately after the 50 star, before the 51st star. For the other two bars, one goes to the left and one to the right. So we have two smaller stars and bars problems, one on each side. Each is 50 stars and 1 bar. Via product rule we get:

$$\binom{51}{1} \cdot \binom{51}{1}$$

■

(d) if there must be more balls in bin A compared to all the other bins?

Solution:

If A must have more than others combined, then $|A| \geq 51$. So there are no bars before star 51. There are 3 bars and 49 remaining stars after that. There are no further requirements of each being non-empty. So we get:

$$\binom{52}{3}$$

■

Problem 3 (Pigeons)

1. **(∗)** What's the minimum number of people required to guarantee that there are two people with the same exam score percentage? *Assume: Percentages are rounded so there are no decimals and the scores are between 0 and 100 (both inclusive).*

   Solution:

   There are a 101 different integers from 0 to 100. As soon as we have 102 people, some two people must share the same score percentage. ■

2. Twenty-five crates of apples are delivered to a store. The apples are of three different sorts, and all the apples in any single crate are of the same sort. Show that among these crates there are at least nine containing the same sort of apple.

[1] There's a slot before each star, except the first star, so 99 such slots.

Solution:

Each crate carries one of three types of apples. $\frac{25}{3} = 8.333\ldots$ So even with the most uniform distribution of crates to apple types, the pigeonhole principle guarantees that there has to be one which has more than $8.333\ldots$ crates. Since crates cannot be fractional, there are 9 crates of some type of apple. ■

3. **(∗)** The combined amount earned by 5 wizard apprentices is 1500 mana coins each week. The new spell book costs 320. Will all of them be able buy this new spell book?

Solution:

The average earning of the apprentices is $\frac{1500}{5} = 300$ coins. If everyone earned 320 coins, then the total earning would be $5 \cdot 320 = 1600$. So there has to be someone who doesn't earn enough to buy the new book.

Another approach: In general, there has to be someone earning ≤ 300 and someone earning ≥ 300. If everyone earned < 300, then the total would be $< 5 \cdot 300 = 1500$ but that isn't the case. On the other hand, if everyone earned > 300, then the total would be $> 5 \cdot 300 = 1500$ but that isn't the case either. So there is someone who earns at most 300 coins and can't buy the spell book.

■

4. You have 6 machines, each has a 6-core processor which can run 6 processes in parallel. When client jobs come to you at the start of the day, you need to assign them to one of these machines: 1 job uses exactly one core. If a job isn't able to start immediately, then the job doesn't finish by end of day and you lose that client because their work didn't get done. If any of the machines has less than 5 jobs running, then it is considered a waste of electricity since your profits won't make up for the cost of keeping these machines running. For each of these days, determine whether jobs are dropped (client lost), things run fine, or if machines are left idle (electricity wasted). You can assume the most optimal assignment of jobs to machines is used each day.

Day 1	Day 2	Day 3	Day 4	Day 5
30	39	25	32	36

Table 1: Table mapping each day to number of jobs.

Solution:

If the number of jobs is 36, then each of the 6-cores in each of the machine's processors is being fully utilized. If there are more, then whatever the optimal assignment, one job has to be dropped.

If the number of jobs is less than 30 jobs, then the average number of jobs per machine is $\leq \frac{29}{6}$. So there has to be a machine with $\leq \frac{29}{6} < 5$ jobs on it by the general pigeonhole principle.

So < 30 is idle/electricity wasted, $\{30, 31, \ldots, 36\}$ is ideal usage and anything ≥ 37 is job dropped/client lost. Filling the table we get:

Day 1	Day 2	Day 3	Day 4	Day 5
30	39	25	32	36
Ideal load	Job drop/Client lost	Machine idle/Electricity wasted	Ideal load	Ideal load

Table 2: Table mapping each day to number of jobs. Bottom row characterizes the job load for the specific situation.

■

5. You have been chosen to play a game against Aliens, the winners keep control over Earth. The challenge/wager is a mathematical game with the following rules:

   - Each box in a 3×3 arrangement of boxes is filled with one of the numbers $-1, 0, 1$ in a totally random way by the Universe (neither you nor the Aliens control it).
   - The sum of numbers in the boxes along the 3 rows and the 3 columns, and the diagonals are considered.
   - If any two sums are equal, you win and Aliens leave forever. If no two sums are equal, Aliens win and you lose Earth forever.

   They explain that depending on what "the diagonals" means, the wager is different. If you only ask for the sum one of the two diagonals to be considered (your choice which), then the wager is only that all resources will belong to the Aliens and humans aren't replaced. If you ask for the sums along both diagonals to be considered, the wager will additionally include humans leaving Earth and renting a barebones settlement on the moon. Further, all humans will have to work for the Aliens in perpetuity, helping with their mining of Earth.

   They claim that a sufficiently advanced species will predict the Universe correctly and make the right choice and hence win the wager. Thankfully for you, you seem to know more discrete mathematics than they do. How will you guarantee a win and avoid losing Earth?[2]

   Solution:

   Most of the story is irrelevant. Once you realize that all possible sums using three of $-1, 0, 1$ (with repetition) are $\{-3, -2, -1, 0, 1, 2, 3\}$, you know that there are only 7 different possible sums. If we consider 3 rows, 3 columns, and also include the 2 diagonals, then we have 8 sums.

   Pigeonhole tells us that two of these 8 sums *must* be the same. See, Discrete Mathematics saves the world from Aliens.[3] ■

Problem 4 (A strong password)

Password strength can (very, very) loosely calculated by computing the logarithm to the base 2 of the number of possible passwords someone has to try to guess a specific password. For example, if all possible passwords are bitstrings of length 3, then there are 8 possibilities and $\log_2 8 = 3$ gives a "strength score" of 3. Compare to all 4-digit card PINs, which are 10^4 in number we get a "strength score" of $\log_2 10^4 = 4 \log_2 10 > 3$. So 4-digit PINs are stronger.

[2] Clearly, Akshar should only be allowed to make up stories for problems where the time spent reading the elaborate made-up situation won't affect grades.

[3] Akshar's on the 9^{th} page of the solutions, so we'll let him talk about mathematics saving the world.

Note 1: This is a very crude measure. Note 2: You should use a calculator to do the logarithm calculation; focus on finding the number of possible passwords of a particular type.

Find the "strength score" for each of these types of passwords. Which type of password is the strongest according to this score.

(a) **(∗)** Length 10. Only lowercase letters.

Solution:

26^{10} with "strength score" of $\log_{10}(26^{10}) \approx 14.15$. ■

(b) Length 9. Only letters, but both lowercase and uppercase allowed.

Solution:

52^9 with "strength score" of $\log_{10}(52^9) \approx 15.44$. ■

(c) Length 8. Any letter and any digit allowed.

Solution:

62^8 with "strength score" of $\log_{10}(62^8) \approx 11.32$. ■

(d) **(∗)** Length 12. Any letter and any digit allowed, as well as any (of 10) punctuations.

Solution:

72^{12} with "strength score" of $\log_{10}(72^{12}) \approx 22.29$. ■

(e) **(∗)** Length 12. At least one uppercase (lowercase and digits allowed, no punctuations).

Solution:

All possibilities without those which have no uppercase (only lowercase + digits). $62^{12} - 36^{12}$ with "strength score" of $\log_{10}(62^{12} - 36^{12}) \approx 21.5$.[4] ■

(f) Length 15. At least one uppercase and at least one digit (lowercase allowed, no punctuations).

Solution:

From the total possibilities we subtract those which have digits but no uppercase and those which have uppercase but no digits. However, in this process we've subtracted those which only have lowercase twice, so we add that back in. This is exact same reasoning as the principle of inclusion-exclusion. $62^{15} - 36^{15} - 52^{15} + 26^{15}$ with "strength score" of $\log_{10}(62^{15} - 36^{15} - 52^{15} + 26^{15}) \approx 26.85$. ■

(g) **(∗)** Length 8–12. Only letters.

[4] Saying at least one uppercase, actually brings the thing closer to 62^{12} rather than to 36^{12} (no uppercase situation). So while there's a subtraction, this is still better than 36^{12} which has a strength of approximately 18.68.

Solution:

$52^8+52^9+52^{10}+52^{11}+52^{12}$ with "strength score" of $\log_{10}\left(52^8+52^9+52^{10}+52^{11}+52^{12}\right) \approx 20.6$.

■

(h) **(∗)** Length 20. Only lowercase letters and the space character.

Solution:

27^{20} with "strength score" of $\log_{10}\left(27^{20}\right) \approx 28.62$. This is the strongest score we've found and that's without uppercase or digits or punctuations. In general, increasing the length is a much faster/better way to improve password strength. ■

(∗) What length password consisting of only the 10 digits will be stronger than the strongest scoring password type above?

Solution:

We need 10^n to have a higher score than 28.62. Solving for $\log_{10}\left(10^n\right) > 28.62$ gives us $n > 28.62$. So a 29 digit PIN is stronger than all the above shenanigans.

This is why, a simple sentence with just lowercase, spaces, and maybe a couple of punctuations can have much stronger security than a make up jumble of letters, digits and punctuations. Altogether, let's assume that's a total of 30 possibilities per slot and n slots. What n beats a score of 50? Solving the equation like earlier gives us: $n = 34$ which has a strength of approximately 50.22.

So a phrase like: "This simple sentence is a decent password." is likely to be a stronger password.

■

Remark: *It is always easier to increase password strength by increasing its length than by increasing the complexity. It is even better to just use a long, memorable phrase or sentence than a weird, immemorable, alpha-numerical-punctuational jumble. Even better, use a (good) password manager to track passwords, this way you only really need to remember the password to this password manager.*