CS1800 Discrete Structures

Solutions to Recitations 4

Strange & Varma *Date: October 15 & 16, 2026*

- This recitation handout contains practice problems. You do *not* need to submit anything.
- We do recommend handwritting your solutions to completion. Do ensure that you are aware of how and why each step in the solution exists.
- For best effects, read through the questions before your recitation section and if possible attempt them on your own. This way you can compare your perspective to the TAs during the recitation.
- If you get stuck on a problem, use your recitation section to get help. Of course, you can also use office hours or post on Campuswire!
- The solutions to this recitation will be posted after Friday. Use those to verify your own attempts and for any clarification.

Problem 1 (Poker hands)

We'll use set terminology to talk about playing cards. The suits are the set $Suits = \{S, C, H, D\}$ referring to *S*pades, *C*lubs, *H*earts, and *D*iamonds. The card values are from the set $Values = \{2, 3, 4, \ldots, 9, 10, J, Q, K, A\}$ for the cards 2 through 10 and the face cards $Faces = \{J, Q, K, A\}$.

Poker hands are a set of five (distinct) cards with certain properties.

(a) Each playing card is defined as a pair of its suit and its value. Define the set of all playing cards $Cards$ using set builder notation.

Solution:

$$Cards = \{(s, v) \mid s \in Suits, v \in Values\}$$

■

(b) *Almost* Royal Flush or Face Flush is *almost* the best possible hand[1]; it is described using set builder notation below. Describe, in English what characterizes this hand. Ensure that you understand the role of each term in the set builder notation.

$$FaceFlushes = \left\{ \left\{ (A, s), (K, s), (Q, s), (J, s), (v, s') \right\} \;\middle|\; s \in Suits, (s \neq s') \vee (v \notin Faces) \right\}$$

Solution:

A Face Flush is a hand containing all the face cards, from the same, suit along with another *non-face* card from a *different* suit. This different suit aspect is what the $s \neq s'$ condition in the set builder notation covers. ■

(c) A very strong poker hand is the "Four of a kind" where you have 4 cards of the same value, one from each suit and any other 5th card. Define a set $4oaks$ using set builder notation containing all possible Four of a kind hands.

Solution:

$$4oaks = \left\{ \left\{ (v, S), (v, C), (v, H), (v, D), (v', s) \right\} \;\middle|\; v, v' \in Values, (v \neq v') \vee (s \in Suits) \right\}$$

■

(d) Describe in English the characteristics of the Full House hand defined by the set below. Ensure that you understand the role of each term in the set builder notation.

$$FullHouses = \left\{ \left\{ (v, s_1), (v, s_2), (v, s_3), (v', r_1), (v', r_2) \right\} \;\middle|\; \{v, v'\} \subset Values, \{s_1, s_2, s_3\} \subset Suits, \{r_1, r_2\} \in \mathcal{P}(Suits) \right\}$$

[1] The basic Royal Flush was too boring, so I modified it.

Solution:

A full house has 3 cards of the value v and 2 cards with a different value $v' \neq v$. So Three of a kind and a Pair. A valid hand must have valid collection of cards and cannot have a card repeat. So we want to ensure that s_i and r_j are distinct. This is done via making them subsets of Suits and hence sets which forces distinctness. ■

(e) Straight is a hand, which contains consecutive values of any suits (a run). The "order" is 2, 3, ..., 10, J, Q, K, A. So the values $7, 8, 9, 10, J$ satisfies the requirement for a run and is a straight hand; similarly $4, 5, 6, 7, 8$ is also a straight. There is special straight: $5, 4, 3, 2, A$ which is also considered a run although here 5 is considered the highest value although A is the highest value in general. Note that this is an exception and $3, 2, A, K, Q$ is not a valid run.

First, assume that there is a set *Runs* of all valid runs. Use that to define the set *Straights* of all straight hands in set builder notation. As a bonus, attempt to write the set *Runs* in set builder notation.

Solution:

Let *Runs* contains the runs as ordered lists of 5 elements, a 5-tuple like (a, b, c, d, e). Using that, *Straights* can be defined as:

$$Straights = \left\{ \left\{ (a, s_1), (b, s_2), (c, s_3), (d, s_4), (e, s_5) \right\} \;\middle|\; (a, b, c, d, e) \in Runs, s_1, s_2, s_3, s_4, s_5 \in Suits \right\}$$

In particular, we don't care whether the s_i repeat or don't.

Runs can be written as follows:

$$Runs = \left\{ (k, k+1, k+2, k+3, k+4) \;\middle|\; k \in \{2, 3, 4, 5, 6\} \right\} \bigcup \left\{ (7, 8, 9, 10, J), (8, 9, 10, J, Q), (9, 10, J, Q, K), (10, J, Q, K, A), (5, 4, 3, 2, A) \right\}$$

Note: We could make it cleaner using some functions, but technically we haven't talked about functions too much, so I'm not going to be pedantic just to make this a little cleaner. ■

Problem 2 (PIEs!)

(a) Find the number of subsets of $T = \{1, 2, 3, 4, 5, 6, 7, 8\}$ that are subsets of neither $A = \{1, 2, 3, 4, 5\}$ nor $B = \{4, 5, 6, 7, 8\}$. You can use the fact that the number of subsets of a set of size n is 2^n.[2]

[2] Another reason to know your powers of 2.

Solution:

We want to find the number of subsets of T that are not subsets of A or B (i.e., subsets contained in T but not in $A \cup B$). Hence, the desired number is $|T| - |A \cup B|$.

By PIE, $|A \cup B| = |A| + |B| - |A \cap B|$. Notice that $A \cap B = \{4, 5\}$, so $|A \cap B| = 2^2 = 4$ using the formula from earlier. Both A and B contain $2^5 = 32$ subsets, and T contains $2^8 = 256$ subsets. The final answer is therefore

$$256 - (32 + 32 - 4) = 256 - 60 = \mathbf{196}.$$

■

(b) In a certain class of 100 students, 70 use a tablet to take notes and 50 use a laptop. How many students solely use a tablet for note taking?

Solution:

$|A \cup B| = |A| + |B| - |A \cap B|$ where $A \cup B$ is total students, A is those who use a tablet for note taking and B is those who use a laptop. We want to know $|A \setminus B| = |A| - |A \cap B|$, so first we find out $|A \cap B|$:

$$|A \cup B| = |A| + |B| - |A \cap B| \implies |A \cap B| = |A| + |B| - |A \cup B| = 70 + 50 - 100 = 20$$

Since there are 20 who use both tablets and laptops, the number of students who only use tablets is $|A| - |A \cap B| = 70 - 20 = 50$. ■

(c) Akshar knows Hindi, Gujarati, and Malayalam apart from English. He's asked around and calculated precise statistics[3] of how many students in this semester's Discrete course know these languages. There are 30 students in the course who know at least one of these three languages but some know two or even all three. There are 20 students who know Hindi, 15 who know Gujarati, and 9 who know Malayalam. There are 8 students who know at least two languages. How many students know all three languages?

Solution:

We will use PIE (Principle of Inclusion-Exclusion):

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - \\ |A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1| \\ + |A_1 \cap A_2 \cap A_3|\end{aligned}$$

Here A_1 = set of people who know Hindi, A_2 = set of people who know Gujarati, A_3 = set of people who know Malayalam. Number of people in at least two sets is:

$$|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1| - 2|A_1 \cap A_2 \cap A_3| = 9$$

Note that we applied PIE in the equation above. If we just added the three intersections of two sets, we would've counted the intersection of all three sets 3 times, but we only want that region counted once. So we subtract that region twice. We don't know what the value of

[3] Nope. All made up just for this problem. But I'll be happy to find out actual numbers.

$|A_1 \cap A_2 \cap A_3|$ is but we can plug all the known information into the first PIE formula and solve for this unknown $x = |A_1 \cap A_2 \cap A_3|$.

So, $30 = (20 + 15 + 9) - (9 + 2x) + x$ which gives $x = 5$. So the number of people who know all three languages is 5. ■

Problem 3 (Counting)

Recall: The cardinality of the Cartesian product, $|A \times B| = |A| \times |B|$. In Combinatorics, the general form of this is called the ***Rule of Product or Product Rule****:*

$$|A_1 \times A_2 \times A_3 \times \ldots A_n| = |A_1| \times |A_2| \times |A_3| \times \ldots |A_n|$$

Similarly, recall the PIE (Principle of Inclusion-Exclusion): $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ and its more general expressions for more terms. In particular, when all the intersections are empty, that is A_1, A_2, $\ldots$, A_n are disjoint sets, then this becomes the ***Sum Rule or Rule of Sum****:*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n| \qquad \textbf{iff } |A_i \cap A_j| = 0 \text{ for all } i \neq j$$

(a) You want to dress up (for Halloween?) as a character from one of the TV shows: Game of Thrones, Community, and Sherlock. You have 20 candidates you can dress up as from Game of Thrones, 10 from Community and 8 from Sherlock. How many total different choices do you have for your Halloween dress up.[4]

Solution:

Straight sum rule application: $20 + 10 + 8$. You can pick one of 20 *OR* one of 10 *OR* one of 8. ■

(b) If each person in your 3 person group for CS1800 decides to dress up as a different character, one from each show from the previous subpart, how many total combinations can there be for your groups character choices?

Solution:

Straight product rule application: $20 \cdot 10 \cdot 8$. First person in group picks one of 20 options *AND* then second person picks one of 10 *AND* then last person chooses one of 8 options. ■

(c) How many 10 character passwords exist if you can use the digits $\{0, 1, \ldots, 9\}$, upper case letters $\{A, B, \ldots, Z\}$ and lower case letters $\{a, b, \ldots, z\}$?

Solution:

Sum rule: 10 digits + 26 lower case + 26 upper case = 62 possible candidates for each character of the password.

Product rule: $62 \times 62 \times 10 \text{ times} \times 62 = 62^{10}$. We usually want to leave it this way rather than find a large number. Leaving it in this form allows us to verify the process rather than arithmetic skills. ■

[4] Yes, none of these fits horror/Halloween vibes, but I randomly pulled TV shows off the top of my head.

(d) How many passwords of 8 to 12 characters exist if you only use the digits $\{0, 1, \ldots, 9\}$?
[***Hint**: Consider sets $P_8, P_9, \ldots, P_{12}$ to be the set of 8, 9, ..., 12 character passwords. You need to find $|P_8 \cup P_9 \cup \cdots \cup P_{12}|$. Thankfully, any $P_i \cap P_j$ is empty.*]

Solution:

Using the same process as the previous subpart, we know the number of n character passwords to be 10^n (since this time we only have the 10 digits available). Sum rule tells us that we need to add 10^n for all possible password lengths n. So we get: $10^8 + 10^9 + 10^{10} + 10^{11} + 10^{12}$. ■

(e) How many 10 character passwords using the digits $\{0, 1, \ldots, 9\}$, upper case letters $\{A, B, \ldots, Z\}$ and lower case letters $\{a, b, \ldots, z\}$ exist if you must use at least one digit and at least one upper case letter?
[***Hint**: Remove unwanted thing count instead of attempting to count wanted things.*]

Solution:

Without any restrictions, we have 62^{10} passwords. The number which have no digit is 52^{10} and those that have no upper case is 36^{10}. So what we want is:

$$62^{10} - 52^{10} - 36^{10}$$

Note: It is a longer discussion as to why ensuring digits and upper case makes passwords stronger even though number of possibilites seems to be going down. It includes lots of counting, probability, and even some human psychology. ■

Problem 4 (Why not Venn?)

Is this a true Venn Diagram for four sets? If so, confirm that all possible intersections are representable. Otherwise, point out which intersection is not representatble in the regions of the diagram.

Try labeling each region with the intersection it can be used to represent. For example, $A = A \cap B^C \cap C^C \cap D^C$, $AB = A \cap B \cap C^C \cap D^c$, $ABC = A \cap B \cap C \cap D^C$ and so on.

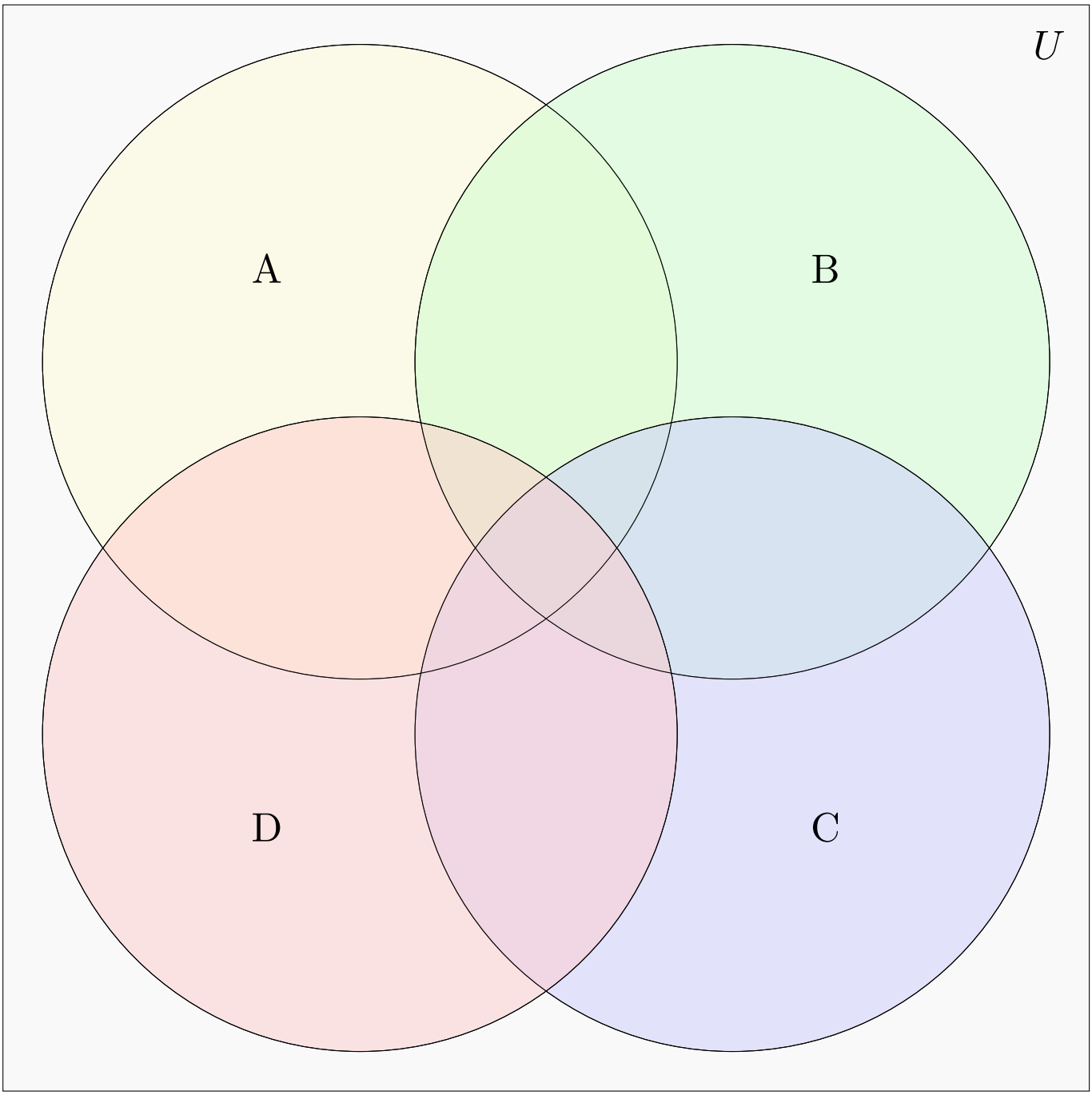


Figure 1: Is this a true Venn Diagram for four sets?

Solution:

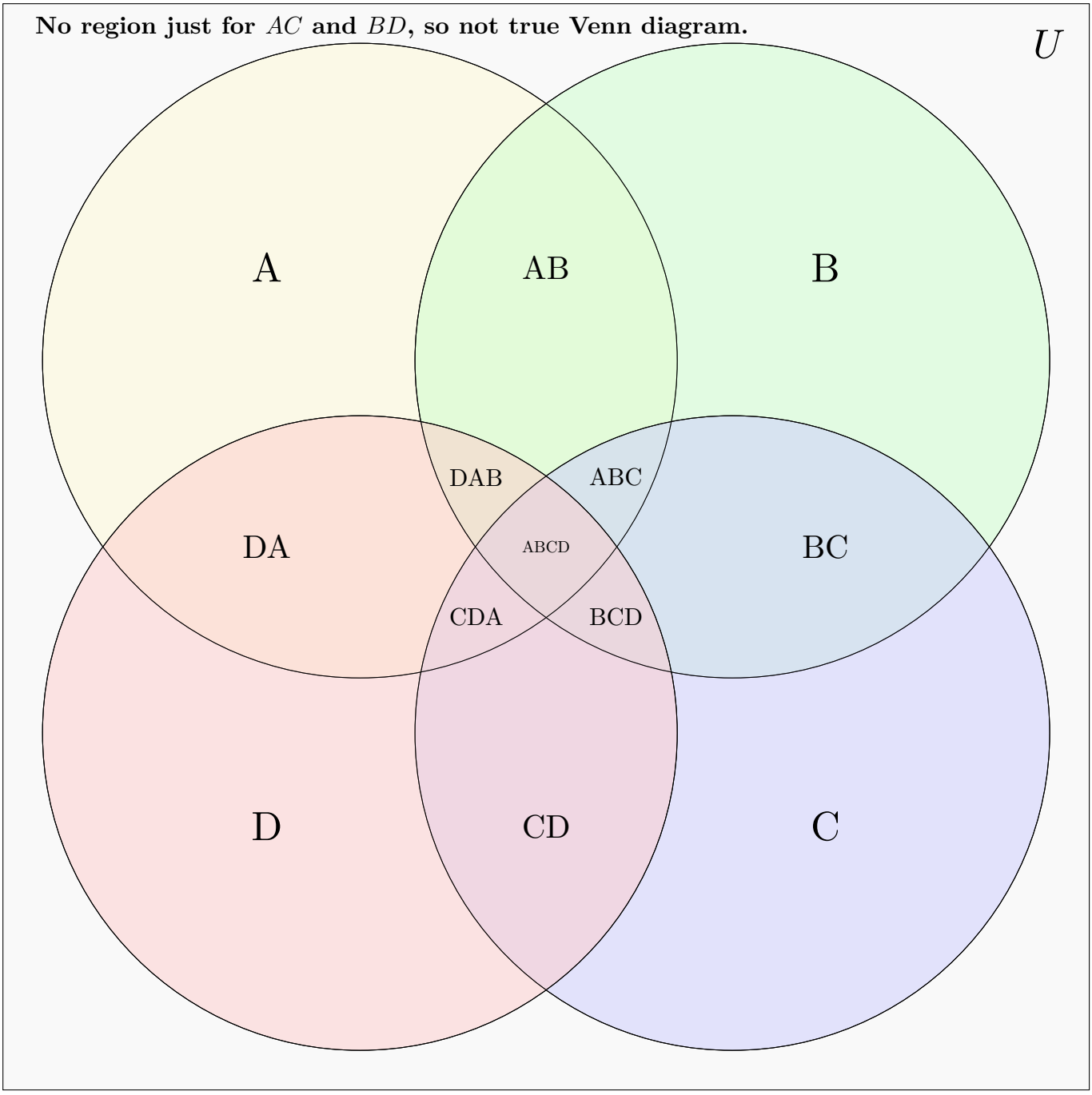


Figure 2: This is not a true Venn diagram for all situations among 4 sets since there is no region for BD or AC. Any time B and D intersect, the region also overlaps with one of A and C. Similarly, whenever A and C overlap the region overlaps with one of B or D.

■

Problem 5 (Is Venn?)

Is this a true Venn Diagram for four sets? If so, confirm that all possible intersections are representable. Otherwise, point out which intersection is not representatble in the regions of the diagram.

Try labeling each region with the intersection it can be used to represent. For example, $A = A \cap B^C \cap C^C \cap D^C$, $AB = A \cap B \cap C^C \cap D^c$, $ABC = A \cap B \cap C \cap D^C$ and so on.

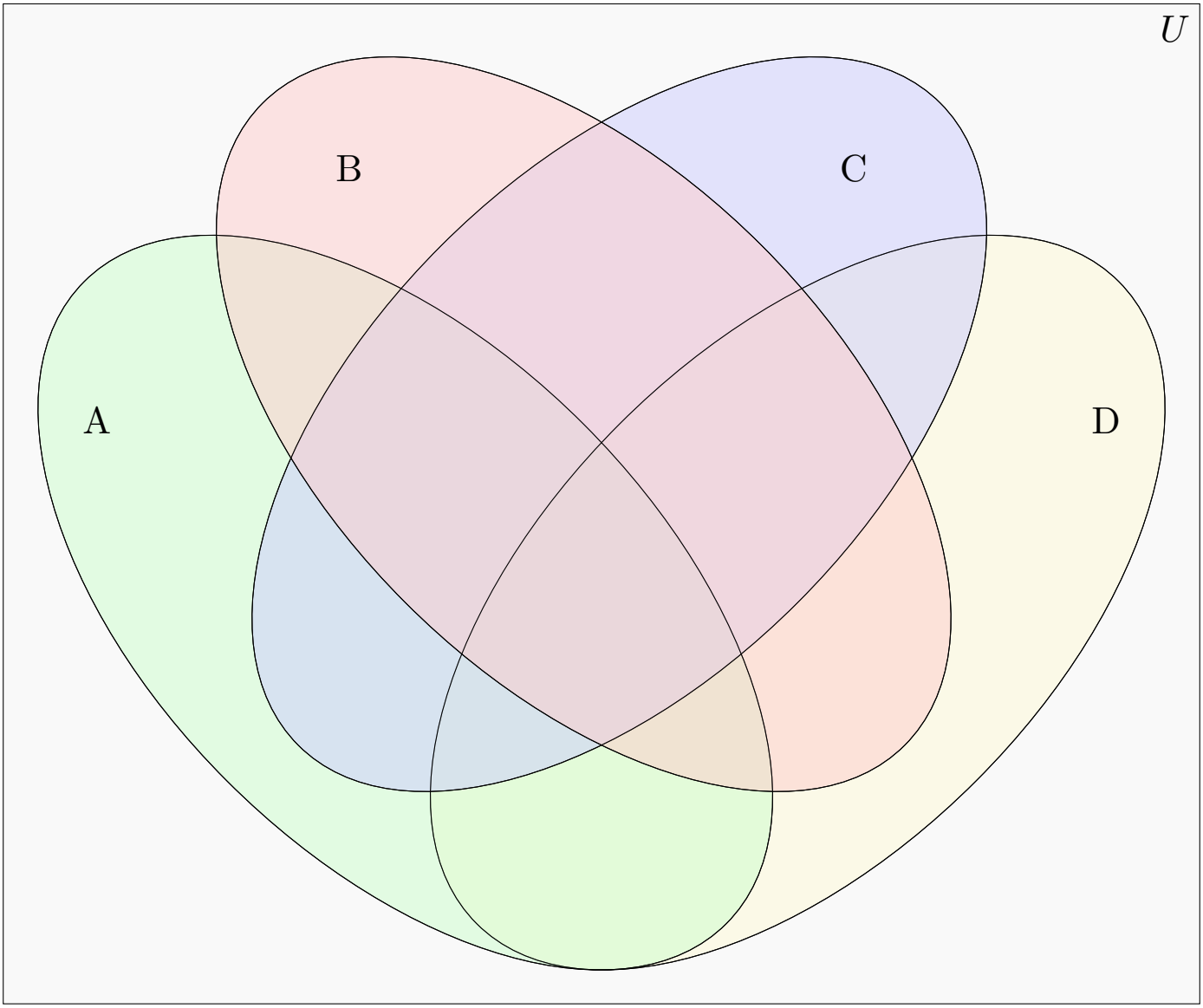


Figure 3: Is this a true Venn Diagram for four sets?

Solution:

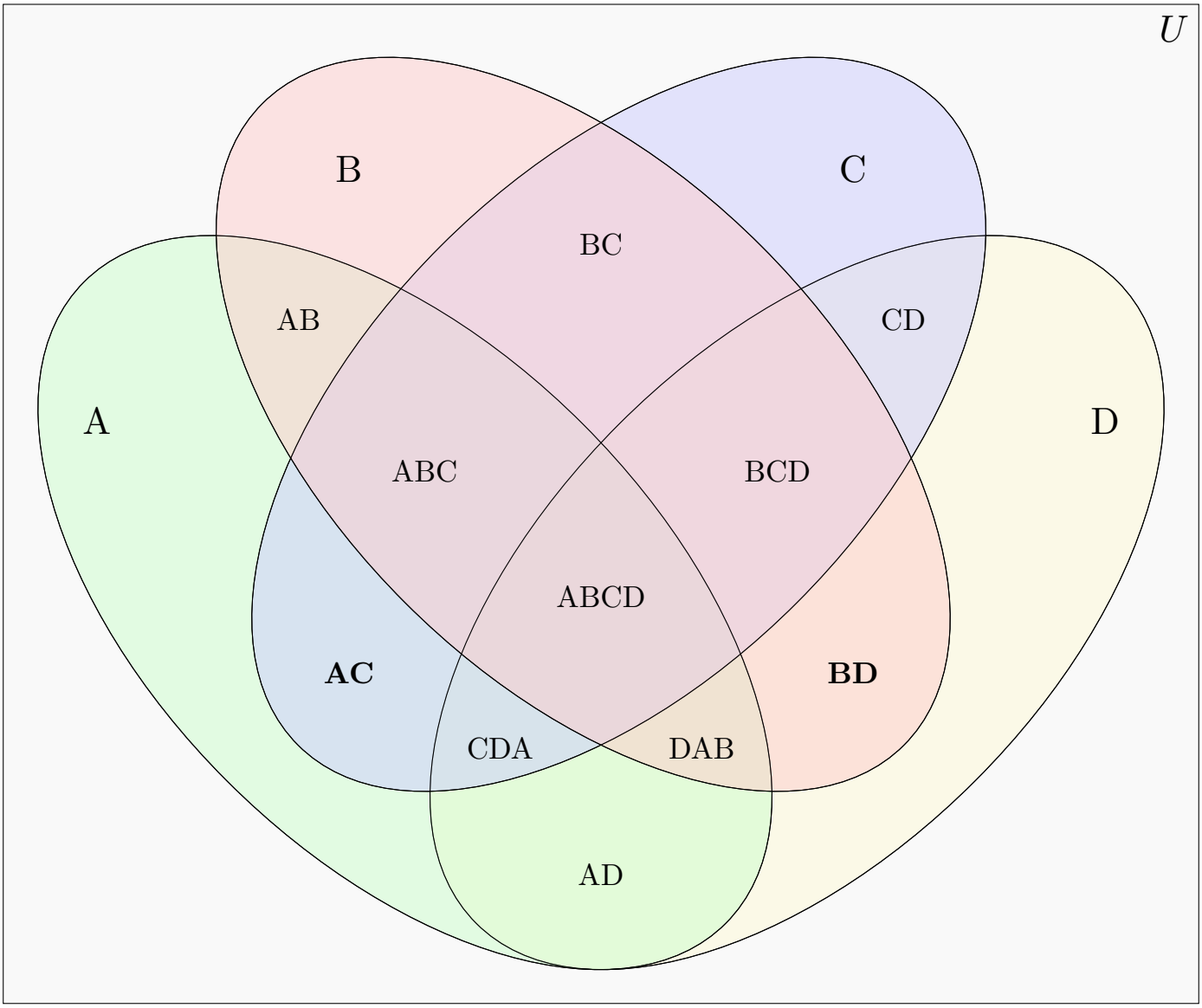


Figure 4: This is a true Venn diagram since all possible intersections among the 4 sets have a dedicated region in the diagram. In particular, the two regions, AC and BD, which couldn't be represented in the diagram in Figure 1 are now representable (bolded in this figure).

■

Problem 6 (Factorio)

Let $X = \{n \in \mathbb{N} \mid 0 \leq n \leq 9\}$. Define $P(x, S)$ to be the predicate $\forall f \in S : x \bmod f = 0$. Here the notation $a \bmod b$ is the remainder when the number a is divided by the number b; so $3 \bmod 2 = 1$, $5 \bmod 3 = 2$, $10 \bmod 5 = 0$. So the predicate is testing that x is divisible by all elements f in the set S.

Define the set:

$$F = \Big\{ (x, S) \in X \times \Big(\mathcal{P}(X) \setminus \{\emptyset\}\Big) \;\Big|\; P(x, S) \wedge \forall R \in \Big(\mathcal{P}(X) \setminus \{S\}\Big) : P(x, R) \implies |R| < |S| \Big\}$$

Write out the set F in roster notation and describe in words what the subset S consists of in the pairs $(x, S) \in F$.

Solution:

F consists of pairs (number, all factors of number). So for each $x \in X$, we find the set S of all factors of it these (x, S) make up F. In roster notation we get:

$$F = \Big\{ \Big(0, \{1\}\Big), \Big(1, \{1\}\Big), \Big(2, \{1, 2\}\Big), \Big(3, \{1, 3\}\Big), \Big(4, \{1, 2, 4\}\Big), \Big(5, \{1, 5\}\Big), \Big(6, \{1, 2, 3, 6\}\Big), \Big(7, \{1, 7\}\Big), \Big(8, \{1, 2, 4, 8\}\Big), \Big(9, \{1, 3, 9\}\Big) \Big\}$$

One important thing to note is that using quantifiers, the set builder notation ensures that only the largest possible set of factors is used in the pairs. While considering subsets to pair with x, apart from ensuring that everything in the subset is a factor of x, it is also ensured that there every other set R satisfying the $P(x, R)$ predicate is smaller than the set S. ■