Professor Hamlin
Day 16

Agenda

1. Admin → 1) Exam 2 Nov 15th
   - Review in Recitation (no quiz)
   - Practice problems this Friday
   - Will cover induction
2. Review
3. Math proofs
4. Induction (weak)

2) Hw 7 is due Nov 15th
- on induction, will have other practice problems to study

Review

Searching algorithms

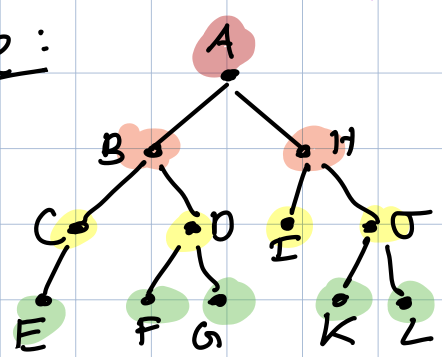
BFS - visit all your neighbors before going on to their neighbors

DFS - get far away as possible before backtracking

Path algorithm

Dijkstra - keep track of cheapest path so far

Exercise:



1) Perform BFS from root

A B H C D I J E F G K L

2) Perform DFS from root

A B C E D F G H I J K L

Why do we care about proofs in CS?

When have you seen proofs before?
Geometry..

But proofs touch everything in CS...

1. Will a program return the correct answer?
   Dijkstra, Euclids, any code
2. Are there problems that are too hard for computers to solve?
   Yes, will your code run forever
3. Is the internet actually secure?
   (My area study!)

Proofs come in different formats (just like essays)

1. Proof of conditional
2. Proof by induction

covered in CS1800

3. Proof by contradiction
4. Proof by reduction

CS3800 Theory of Comp

...

All of them involve using rigorous logic to convince the reader of the proofs correctness

Proving a conditional

"if x then y" - how can we prove this statement true e.g.:

"if shape is a □ then shape is (also) ▭"

First we define:

□ - polygon w/ 4 equal sides and 90° angles

▭ - polygon w/ 4 sides and 90° angles

Proof: Assume shape is a square

(implies) $\Rightarrow$ shape has 4 equal length sides and 90° angles
(by def'n of square)

$\Rightarrow$ shape is rectangle
(by def'n of rectangle)

Generally: $P \to Q$

1. Assume P
2. Give series of implications to get to Q

$P \to \bullet \to \bullet \to \bullet \to Q$

Note we had to use P to get to Q

Exercise: Define

- integer z is even if $\exists$ some integer a s.t $z = 2a$

Prove: If z is even, then z^2 is also even

(useful fact: $a, b \in \mathbb{Z}$ then $c \in \mathbb{Z}$ where $c = ab$)

Assume z is even

$\Rightarrow z = 2a$ (by def'n of even)

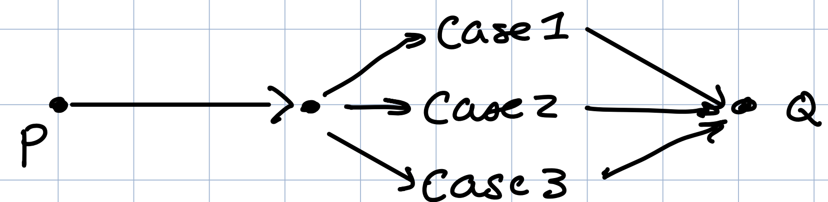
$\Rightarrow z^2 = (2a)^2$

$\Rightarrow z^2 = 4a^2$

$\Rightarrow z^2 = 2(\underbrace{2a^2}_{\text{is int. by useful fact}})$

Useful Proof Trick: Cases

Just like counting can break proofs into cases



Break up into disjoint cases and argue individually

Example If wear sun screen on every sunny day then won't get a sunburn.

Proof Case 1: sunny $\rightarrow$ wear sunscreen $\rightarrow$ no burn

Case 2: no sun $\rightarrow$ no burn

Useful Proof Trick: Without Loss of Generality
(WLOG)

When using cases, sometimes cases are very similar

Can argue these cases all at once (WLOG)

Example if you cut a 100g cheese block into 2 pieces, one side will be at least 50g

Proof Assume we cut block in two
WLOG, larger piece is A, smaller is B
$(A \geq B)$

$\Rightarrow 100 = A + B$

$\Rightarrow 100 \leq A + A$ (by $B \leq A$)

$\Rightarrow 100 \leq 2A$

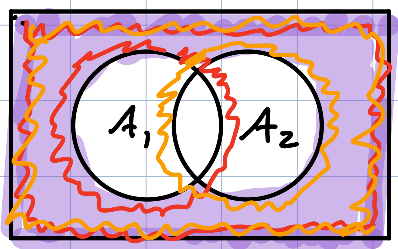
$\Rightarrow 50 \leq A$

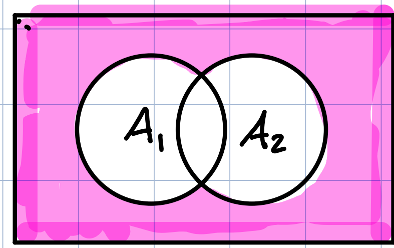
if not wlog

$\boxed{\begin{array}{l} \text{A is small, than B} \\ A \leq B \\ 100 = A+B \\ 100 \leq 2B \\ 50 \leq B \end{array}}$

Remember De Morgan's for sets....

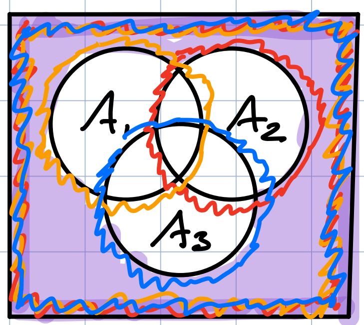
$$\overline{A_1} \cap \overline{A_2} = \overline{A_1 \cup A_2}$$

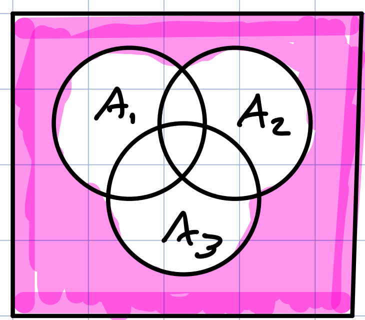




Does this hold for 3 sets?

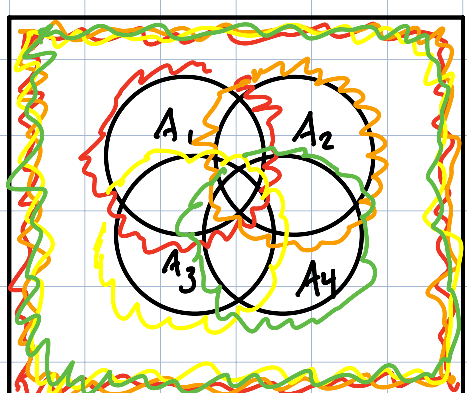
$$\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} = \overline{A_1 \cup A_2 \cup A_3}$$



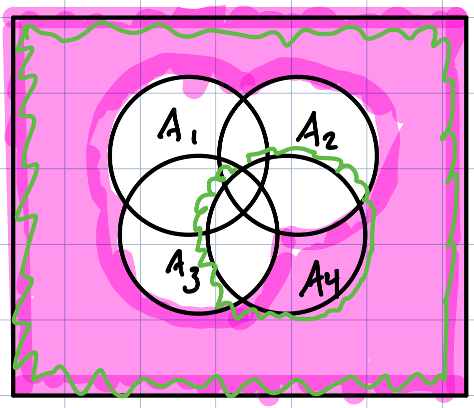


What about 4 sets?

$$\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} = \overline{A_1 \cup A_2 \cup A_3 \cup A_4}$$

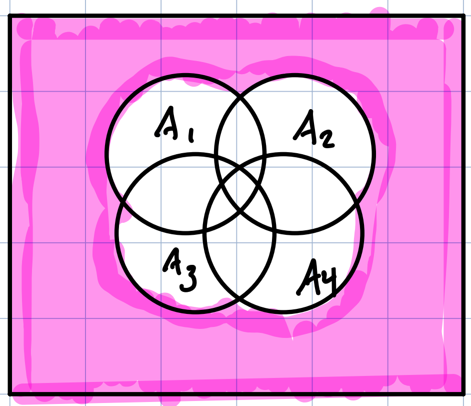


This is starting to get messy. Can we do this a neater way?



Know $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 = \overline{A_1 \cup A_2 \cup A_3}$ by previous section so

$$\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4 = (\overline{A_1 \cup A_2 \cup A_3}) \cap \bar{A}_4$$



This is much easier to show by using our work from previous step

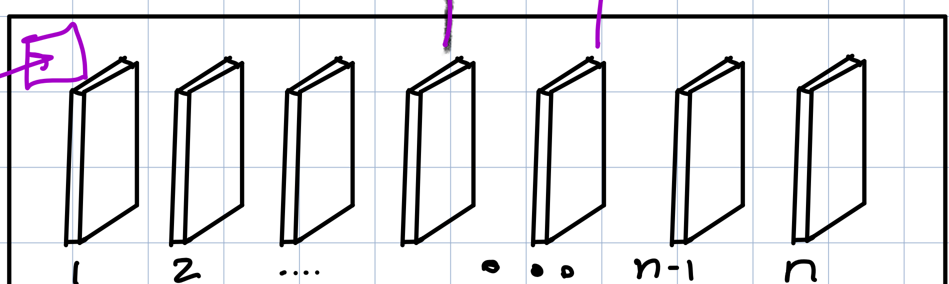
So what about n sets?

$$\bar{A}_1 \cap \dots \cap \bar{A}_n = A_1 \cup \dots \cup A_n$$

Could prove it individually for each $n = 1, 2, 3, \dots$ but we can always use our work from $n-1$ to prove n.

this will fall, then will hold for any pair of dominos

base case



Dominoes! Push one over the rest follow.

Induction (weak)

Process: 1) prove first statement
2) Show that statement for $n-1$ implies n
(WLOG)

Example | I want to know that my candy in my bag is all the same w/ two rules

1) Can only take and examine 1 piece of candy from the bag
2) Can only say one sentence to the person on your right

I will only talk to the last person → how can you convince me?

Anatomy of induction proof

Show the sum of first n odd numbers is n^2.

Proof | We wish to show:

$$1 + 3 + 5 + 7 + \dots + (2n-1) = n^2$$

Define the problem formally in terms of n

Base case: $n = 1$ thus $1 = 1^2$

Choose base case value and show statement holds

"if statement holds for n then holds for n+1"

Inductive Step: if it holds for $n = t$ e.g.

if $P \Rightarrow Q$ $P\ (1+3+5+\ldots+(2t-1) = t^2)$ Inductive Hypoth

then it holds for $n = t+1$ e.g:

$Q\ (1+3+5+\ldots+(2t-1)+(2(t+1)-1) = (t+1)^2)$

State the inductive hypothesis "if holds for n then it holds for n+1"

Starting with

$$1+3+5+\ldots+(2t-1)+(2(t+1)-1)$$

$$\Rightarrow 1+3+5+\ldots+(2t-1)+(2t+1)$$

$$\Rightarrow t^2+(2t+1)$$ by assumption in I.H.

$$\Rightarrow t^2+2t+1$$

$$\Rightarrow (t+1)^2$$ // done!

Write statement for n+1 then start manipulating one side to get it to equal the other.

Hint: you will <u>always</u> need to use I.H.

Another take (w/o commentary)

Inductive Step: if $\underbrace{\text{statement } n}_{P}$, then $\underbrace{n+1}_{Q}$

→ Conditional proof

$P = 1 + 3 + 5 + \ldots + (2t-1) = t^2$

$Q = 1 + 3 + 5 + \ldots + (2t-1) + (2(t+1)-1) = (t+1)^2$

⇒ Assume P, for $n = t$

$1 + 3 + 5 + \ldots + (2t-1) = t^2$

start $1 + 3 + 5 + \ldots + (2t-1) + (2(t+1)-1)$

$1 + 3 + 5 + \ldots + (2t-1) + (2t+1)$ (by I.H. (the thing we assumed))

$t^2 + (2t+1)$

$(t+1)^2$ ✓

1 Induction Example (Recipe & Rubric)

Our purpose here is to emphasize a recipe for how to approach writing an induction proof. On first look, it may appear a bit strict in requiring that certain steps be shown. In previous semesters, I've been a bit more relaxed with the formatting requirements and I've found that many students can lose track of precisely where they are in the proof, causing confusion. I hope that this recipe & rubric[1] help structure everyone's induction proofs so they their studies are more productive.

The green rubric boxes are immediately below the portion of the solution they refer to.

2 Geometric Series

Using induction, prove the following geometric series formula:

Let r be a real number not equal to zero. Then, for any natural number n greater than or equal to 1:

$$\sum_{i=1}^{n} a_1 r^{i-1} = a_1 \frac{1-r^n}{1-r}$$

Solution

Statement n is $\sum_{i=1}^{n} a_1 r^{i-1} = a_1 \frac{1-r^n}{1-r}$

Rubric

1 point: clearly writing what statement n is
(Students sometimes lose track of what they're proving, especially when writing induction proofs seperately from the problem statement. Labelling this right up top is helpful.)

Base Case (statement 1):

$$\sum_{i=1}^{1} ar^{i-1} = a_1 = a_1 \frac{1-r}{1-r}$$

Rubric

1 point: providing a clear base case index (e.g. here n=1) and writing what statement n=1 is
1 point: demonstration that the base case is true.

[1] Please note that the rubric point values here are suggestions only, different problems may weight things differently. We include them to give a rough sense of what that future rubric will be.

statement interms of n

if P then Q → statement in terms of n+1

Inductive Step: "if statement n then statement n + 1"

Rubric

1 point: writing this label for the inductive step, you're welcome to write it exactly as "Inductive Step: if statement n then statement n + 1"
Feel free to write $S(n)$ instead of statement n if you prefer, though I worry that this more algebraic notation allows students to forget that its just a statement, you couldn't multiply it by 2 to get $2S(n)$.
(These are easy points to earn, and writing this title is a helpful reminder to reader and author alike: what follows is a proof of $S(n) \to S(n+1)$.)

Assume statement n is true, that is: Assume P, write out P

Inductive hypothesis

$$\left[\sum_{i=1}^{n} a_1 r^{i-1} = a_1 \frac{1-r^n}{1-r}\right]$$

Rubric

1 point: writing out statement n (the inductive hypothesis), explicitly.

write out Q

start w/ 2 halves of Q

math

Then:

$$\begin{aligned}
\sum_{i=1}^{n+1} a_1 r^{i-1} &= \sum_{i=1}^{n} a_1 r^{i-1} + a_1 r^n \\
&= a_1 \frac{1-r^n}{1-r} + a_1 r^n \\
&= a_1 \frac{r^n * (1-r) + 1 - r^n}{1-r} \\
&= a_1 \frac{r^n - r^{n+1} + 1 - r^n}{1-r} \\
&= a_1 \frac{1-r^{n+1}}{1-r}
\end{aligned}$$

Rubric

- 1 point: correctly writing the half of statement n + 1 (e.g. $\sum_{i=1}^{n+1} a_1 r^{i-1}$ above)
- 1 point: correctly writing the other half of statement n + 1 (e.g. $a_1 \frac{1-r^{n+1}}{1-r}$)
- 2 point: applying the inductive hypothesis (statement n) correctly within the reasoning
- 2 point: reasoning / algebra "glue" to get between either side of statement n + 1

Induction Tip

This last part is often the most challenging for students. Sometimes its the algebra which is tough and other times students are attempting to prove something which isn't true because they've made a mistake in the induction structure! Here's a few tips on setting up the induction structure so you can isolate your algebra challenges properly:

- Very often I find problems are easier to think about when we start[a] on the summation side of statement n + 1, $\sum_{i=1}^{n+1} a_1 r^{i-1}$, and work our way towards the other, simpler side of things.
- Towards the bottom of your page, write the second (simpler) side of things (i.e. $a_1 \frac{1-r^{n+1}}{1-r}$). Its worth a point and serves to remind us where we're headed.
- If you've got a summation to work from, try popping out that final term in the summation to set up applying our indutive hypothesis (statement n). Here's a silly little summation notation reminder of how that works:

$$\sum_{k=1}^{n+1} k = 1 + 2 + 3 + 4 + \ldots + n + (n+1) = (\sum_{k=1}^{n} k) + (n+1)$$

- Your reasoning must should[b] the inductive hypothesis (statement n), be on the lookout for a place to apply that assumption!

Following these four tips above yields the equalities below:

$$\begin{aligned}
\sum_{i=1}^{n+1} a_1 r^{i-1} &= \sum_{i=1}^{n} a_1 r^{i-1} + a_1 r^n \\
&= a_1 \frac{1-r^n}{1-r} + a_1 r^n \\
\ldots \\
&= a_1 \frac{1-r^{n+1}}{1-r}
\end{aligned}$$

Now that we've settled our induction structure, you can focus on the algebra required to fill in the ... above. (And, if the worst comes to it, know that you've scored the majority of the points on this induction proof ... students who struggle to connect the dots here might have an algebra challenge but their induction skills are solid).

[a] To be clear, you can right a correct induction proof, earning full credit, starting from either side of an equality / inequality. Working from complex (summation) to simple often helps students though.

[b] Should you not use it, then you've got a proof of statement n + 1 which doesn't rely on statement n. It may be a valid proof but its not an induction proof!

Exercise 1 Prove using induction that the sum of the first n integers is $\frac{n(n+1)}{2}$

1. define & remind: statement n
2. choose base case n & show it
3. write "inductive step: if statement n then statement n + 1
4. Prove inductive step:
   a. assume statement n (inductive hypothesis)
   b. write statement n + 1 in two halves
      (tip: start at sum side, work towards other side)
   c. apply assumption to get from one half to other

We want to show:

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

Base case: $n=1$

$$1 = \frac{1(1+1)}{2}$$

$$1 = 1 \checkmark$$

Inductive: if $\underbrace{\text{statement } n}_{P}$, then $\underbrace{\text{statement } n+1}_{Q}$

$$P = \text{"}1+2+3+\dots+n = \frac{n(n+1)}{2}\text{"}$$

$$Q = \text{"}1+2+\dots+n+(n+1) = \frac{(n+1)(n+2)}{2}\text{"}$$

Assume $P = 1+2+\dots+n = \frac{n(n+1)}{2}$

start with

$$\Rightarrow 1+2+\dots+n+n+1$$

$$\frac{n(n+1)}{2} + n+1\left(\frac{2}{2}\right) \quad \text{(by I.H.)}$$

$$\frac{n^2+n+2n+2}{2}$$

$$\frac{n^2+3n+2}{2}$$

$$\frac{(n+1)(n+2)}{2}$$

✓