CS1800 Day 16

(we'll start @ 952 to give everyone a chance to get a seat)

Admin:
- HW6 (graphs) due this Friday Nov 8
- Exam2:
    - next friday Nov 15
    - review next week in recitation (no quiz)
    - you'll get practice problems this Friday Nov 8
- HW7 (induction) is also due next Friday Nov 15
    - not graded before exam2 but you'll have other induction examples to study from

Content:

Induction (proving a sequence of statements)
- Proving a conditional P -> Q
- Weak Induction (algebraic equality)

Why write a proof in CS?

So far:
To demonstrate that an algorithm works
- e.g. how can we show that Dijkstra's Algorithm really does find the shortest path?
  (our previous class notes are suggestive ... but I wouldn't call them a "proof")
- e.g. how can we show that Euclid's Division Algorithm really does convert to binary?

Towards the end of the semester
To demonstrate that one function is always bigger than another
- e.g. *insertion sort will take more operations to sort a list than merge sort

* ... in the worst case (more to come later)

Proving a conditional

Define:

a rectangle is a polygon with 4 sides whose interior angles are all 90 degrees

a square is a polygon with 4 equal sides whose interior angles are all 90 degrees

Example conditional statement:

If a shape is a square, then that shape is (also) a rectangle

Proof:

Assume a shape is a square

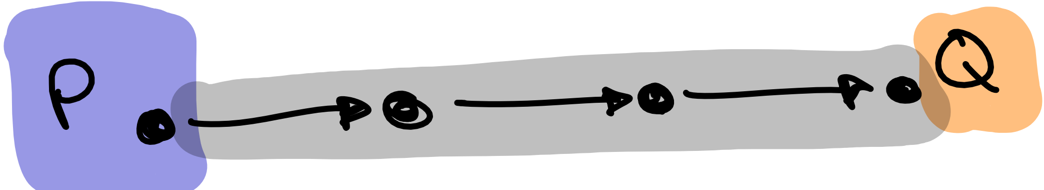
- then the shape has 4 sides and interior angles which are all 90 degrees
- then the shape is a rectangle

Proving a conditional

$P \to Q$

① Assume P

② Give sequence of implications which end at Q



Tip: Use P somewhere in your argument to get to Q

(Otherwise Q true by itself, if so its simpler to drop conditioning on P)

In Class Activity:

Define: integer z is even if there exists some integer a with z = 2a

Useful fact: multiplying two integers always yields another integer

Prove the following statement:

If an integer z is even, then z squared is also even.

ASSUME z IS EVEN $\Rightarrow \exists\, a \in \mathbb{Z}$

$z = 10$

$z^2 = 100 = 50 \cdot 2$

$z = 2a$

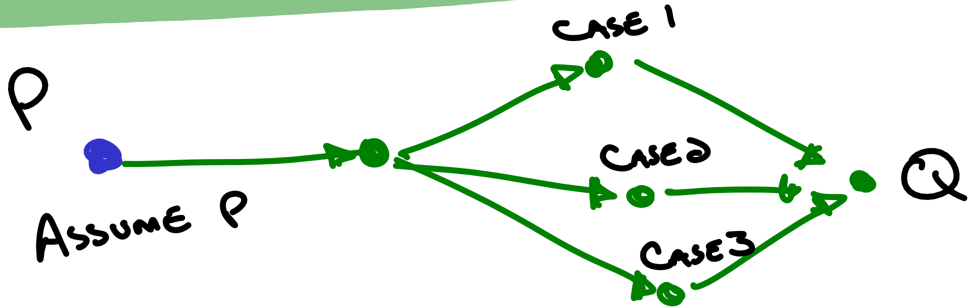
$z^2 = 4a^2$

$= 2(2a^2)$

$2a^2$ IS AN INTEGER

SO z^2 IS ALSO EVEN

Proof Move: Break Your Argument Into Cases



Approach:

Partition all possibilities into cases, argue each will imply Q

Example: If you wear sunscreen on every sunny day, then you won't get any burns from the sun.

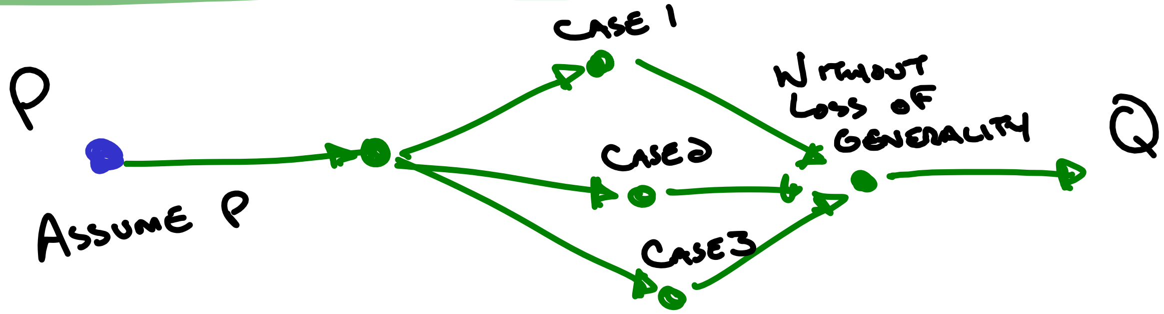
Proof: Assume if one is in the sun, then they wear sunscreen

case1: sunny day → one wears sunscreen → no burn from the sun

case2: not a sunny day → no burn from the sun

Notice: we argue from each case to conclusion Q. The argument has a feeling of "no matter what case ... Q"

Proof Move: Without Loss of Generality (WLOG)



Approach:

Simplify your argument by combining your cases, often by re-labelling or re-orienting how you define things.

Example: If you cut a 100g wheel of cheese into 2 pieces, one side will be at least 50g

Proof: Assume we cut a 100g wheel of cheese into two pieces.

WLOG, let us call the mass of larger piece A and the smaller mass B.

Then 100 $= A + B$

$\leq A + A = 2A$ (first equality from $B \leq A$)

so that $50 <= A$

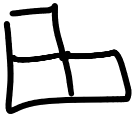
Tiling a Bathroom Floor

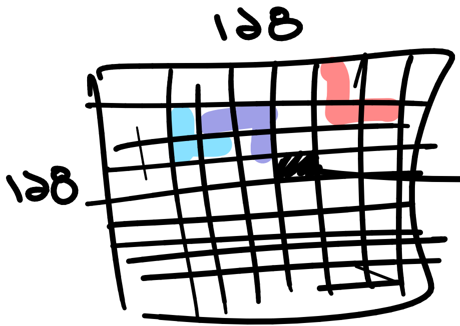
$n \in \mathbb{N}$
$n \neq 0$

Given a Bathroom Floor is an array of $2^n \times 2^n$

May we always be able to tile all but one space with an "L shape"?

→ No matter where spot is





→ Spot

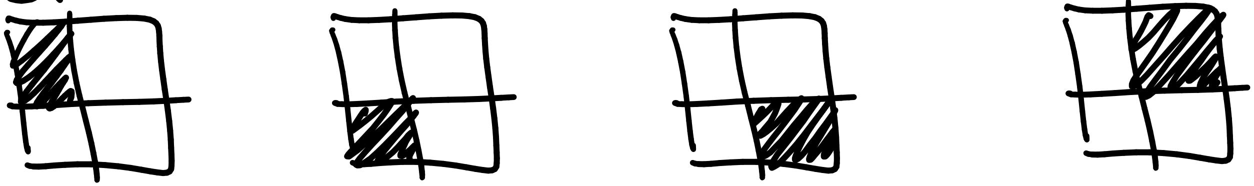
LET'S SOLVE SIMPLER PROBLEM . . .

STATEMENT ($n=1$)

A $2^1 \times 2^1$ ARRAY WITH ONE MISSING SPOT MAY BE TILED WITH L SHAPES

PROOF

CASE 0 CASE 1 CASE 2 CASE 3



IN ANY CASE, WE MAY TILE THE ARRAY

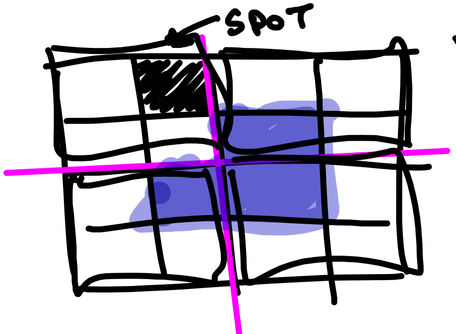
GENERALIZING FROM $n=1$ TO $n=2$

STATEMENT ($n=2$)

A $2^2 \times 2^2$ ARRAY WITH ONE MISSING SPOT MAY BE TILED WITH L SHAPES

PROOF

WLOG: ASSUME MISSING SPOT IN TOP LEFT QUADRANT (ROTATE ARRAY UNTIL THIS IS TRUE)



- ADD L TILE TO INDUCE "SPOT" IN OTHER QUADRANTS
- $n=1$ CASE (PREVIOUS SLIDE) TELLS US EACH QUADRANT CAN BE TILED

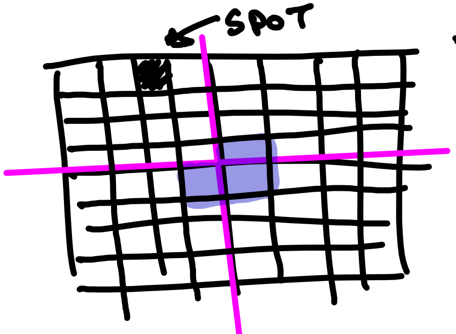
GENERALIZING FROM $n=2$ TO $n=3$

STATEMENT ($n=3$)

A $2^3 \times 2^3$ ARRAY WITH ONE MISSING SPOT MAY BE TILED WITH L SHAPES

PROOF

WLOG: ASSUME MISSING SPOT IN TOP LEFT QUADRANT (ROTATE ARRAY UNTIL THIS IS TRUE)



- ADD L TILE TO INDUCE "SPOT" IN OTHER QUADRANTS
- $n=2$ CASE (PREVIOUS SLIDE) TELLS US EACH QUADRANT CAN BE TILED

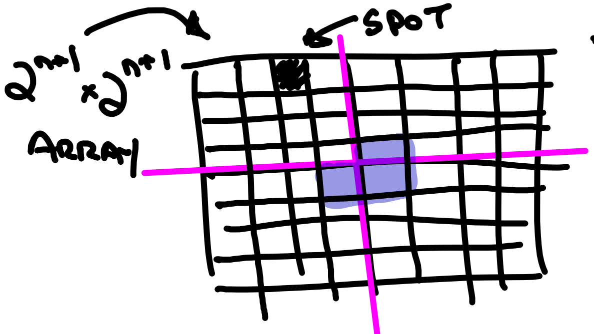
GENERALIZING FROM n TO $n+1$

STATEMENT (n)

A $2^n \times 2^n$ ARRAY WITH ONE MISSING SPOT MAY BE TILED WITH L SHAPES

PROOF

WLOG: ASSUME MISSING SPOT IN TOP LEFT QUADRANT (ROTATE ARRAY UNTIL THIS IS TRUE)



- ADD L TILE TO INDUCE "SPOT" IN OTHER QUADRANTS
- n CASE (PREVIOUS SLIDE) TELLS US EACH QUADRANT CAN BE TILED

Induction (Weak):

Induction allows us to prove a never-ending sequence of statements: S(1), S(2), S(3), S(4), ...

Process:

- Prove the first statement, S(n) for some n

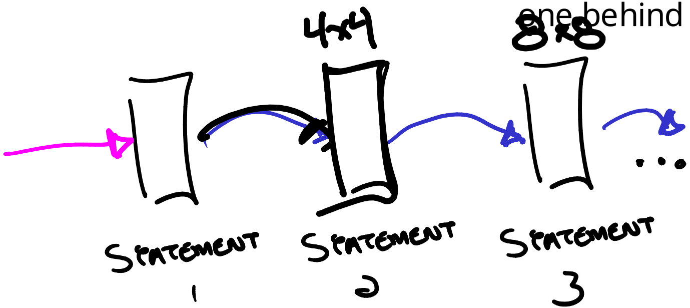
- Show that each statement implies the next statement:

Metaphor (Dominos):

To knock over all the dominos

- Push over the first one

- Place each other domino so that if the one behind it falls, it too will fall



A delicious induction example

IF $z_1\ z_2$ ARE EVEN THEN $z_1 \cdot z_2$ IS EVEN

ASSUME z_1 IS EVEN $\Rightarrow \exists\, a_1 \in \mathbb{Z}\ \ z_1 = 2a_1$

z_2 IS EVEN $\Rightarrow \exists\, a_2 \in \mathbb{Z}\ \ z_2 = 2a_2$

$$z_1 \cdot z_2 = 2 \cdot \underbrace{(2 a_1 a_2)}_{\text{INTEGER}}$$

$z_1 z_2$ IS EVEN

Weak Induction: Algebraic Equality

Show that the sum of the first n odd numbers is n^2

$$1+3+5+7+\ldots+(2n-1)=n^2 \qquad \forall n\in\mathbb{N} \quad n\geq 1$$

While not part of our proof, it can help to test statement out a bit. Is this statement reasonable? We explore below:

$n=1$ CASE	$n=2$ CASE	$n=5$ CASE
BIGGEST TERM: $2\cdot 1-1=1$	BIGGEST TERM: $2\cdot 2-1=3$	BIGGEST TERM: $2\cdot 5-1=9$
$1\overset{?}{=}1^2$	$1+3\overset{?}{=}2^2$	$1+3+5+7+9\overset{?}{=}5^2$

A template for equality / inequality style induction proofs:

1. define & remind: statement n
2. choose base case n & show it
3. write "inductive step: if statement n then statement n + 1"
4. Prove inductive step:
    a. assume statement n (inductive hypothesis)
    b. write statement n + 1 in two halves
        (tip: start at sum side, work towards other side)
    c. apply assumption to get from one half to other

STATEMENT N:

$$1+3+5+7+\dots+(2n-1)=n^2$$

BASE CASE N=1

$$N=1=1^2=N^2$$

INDUCTIVE STEP: $S(n) \rightarrow S(n+1)$

ASSUME $1+3+5+7+\dots+(2n-1)=n^2$

$$1+3+\dots+(2n-1)+(2(n+1)-1)=n^2+2(n+1)-1$$
$$=n^2+2n+2-1$$
$$=n^2+2n+1$$
$$=(n+1)^2$$

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STATEMENT n

$1+3+5+\ldots+(2n-1)=n^2$

BASE CASE $(n=1)$:

$1=1^2=n^2$

INDUCTIVE STEP: IF $S(n) \to S(n+1)$

ASSUME $1+3+5+\ldots+(2n-1)=n^2$

INDUCTIVE HYPOTHESIS

$1+3+5+\ldots+(2n-1)+(2(n+1)-1)=n^2+2(n+1)-1$

$=n^2+2n+1$

$=(n+1)^2$

<walk through the induction rubric / guide on website>

In Class Activity:

Using induction prove that the sum of consecutive integers is n (n+1) / 2:

STATEMENT N : $1+2+3+4+\dots+n = \frac{n(n+1)}{2}$

BASE CASE N=1 : $1 = \frac{1(1+1)}{2} = \frac{N(N+1)}{2}$

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In Class Activity:

$$S(n+1): 1+2+3+\ldots+n+(n+1) = \frac{(n+1)((n+1)+1)}{2}$$

Using induction prove that the sum of consecutive integers is n (n+1) / 2:

$$1+2+3+4+\ldots+n = \frac{n(n+1)}{2}$$

INDUCTIVE STEP: $S(n) \to S(n+1)$

ASSUME $1+2+3+4+\ldots+n = \frac{n(n+1)}{2}$

$$1+2+3+\ldots+n+(n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n(n+1)+2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}$$

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BASE CASE N=1

$n=1 = \frac{1(n+1)}{2}$

INDUCTIVE STEP $S(n) \to S(n+1)$

ASSUME $1+2+3+4+\ldots+n = \frac{n(n+1)}{2}$

THEN $1+2+3+4+\ldots+n+(n+1) = \frac{n(n+1)}{2} + (n+1)$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$