CS1800 Day8

Admin:
- regrade requests (https://course.ccs.neu.edu/cs1800/admin_hw.html#regrades)
- hw formatting deal (for hw1's formatting penalties only):
    - we'll reduce hw1 penalties in proportion to HW formatting penalty decrease from hw1 to hw3
        - example:

            hw1 had 30% of class have HW formatting penalty

            hw3 had 15% of class have HW formatting penalty

            we'll cut HW1's formatting penalty in half

Content:
- pigeonhole principle
- product rule
    - set operation: cartesian product of two sets
- principle of inclusion exclusion
    - sum rule

Floor and Ceiling Functions:

Ceiling

$\lceil x \rceil$ = SMALLEST $y \in \mathbb{Z}$ WITH $x \leq y$

"ROUND UP TO NEAREST WHOLE NUMBER"

$\lceil 7.1 \rceil = 8$

$\lceil 7 \rceil = 7$

$\lceil 7.0001 \rceil = 8$

Floor

$\lfloor x \rfloor$ = LARGEST $y \in \mathbb{Z}$ WITH $y \leq x$

"ROUND DOWN TO NEAREST WHOLE NUMBER"

$\lfloor 6.2 \rfloor = 6$

$\lfloor 5.999 \rfloor = 5$

$\lfloor 6.00001 \rfloor = 6$

In Class Activity (quickly)

$x = \lfloor 1.2 \rfloor$	$x = \lceil 71.2 \rceil$	$x = \lceil 100 \rceil$	$x = \lfloor -1002 \rfloor$
1	72	100	-1002

A strategy to quickly split a deck of cards in half (roughly): split-and-pick

To begin a card game where each player wants the most cards:

- Player 1 splits the deck in approximately half:

- Player 2 chooses which of the two "halves" they'd like to play

Notice: No matter how player1 splits the deck, player 2 can choose a pile with, at least, half of the cards.

Chocolate Pigeonhole

$\frac{11}{3} = 3\,2/3$ $\left\lceil \frac{N}{K} \right\rceil$

Suppose I divide N chocolates into 3 piles.
You may take (and keep) the pile with the most chocolate.

How many chocolates are you gauranteed (at least) to get, no matter how I split?

Pile 1 Pile 2 Pile 3

N chocs	0	1	2	3	4	5	6	7	8	9	10	11
K = gauranteed min chocs in some pile	0	1	1	1	2	2	2	3	3	3	4	4

Pigeonhole Principle

For all ways one divides N items into K piles, there exists a pile with at least ceiling(N/k) items.

$$X = \left\lceil \frac{N}{K} \right\rceil$$

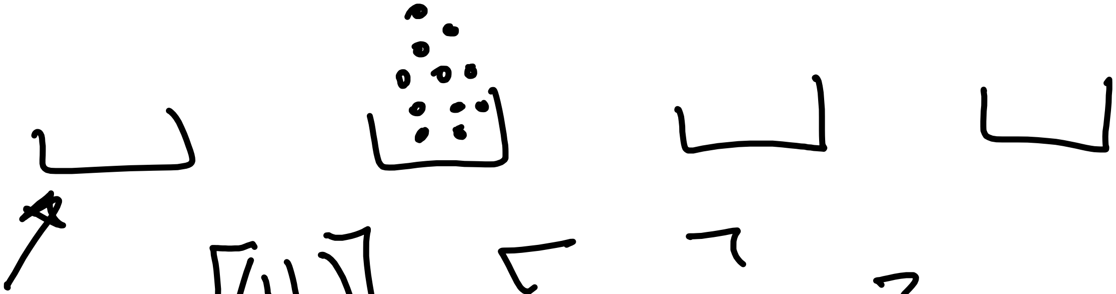
N → # of items

K → # of piles

"there exists a pile with at least x items in it"

"the pile with the most items has," at least, x items in it

11 CHOCOLATES 4 PILES



$$\left\lceil \left(\frac{11}{4}\right) \right\rceil = \lceil 2.75 \rceil = 3$$

In Class Activity: Pigeonhole

If we group 3 pigeons into 2 nests, how many pigeons will be in the nest with the most pigeons?

$$x = \left\lceil \frac{3}{2} \right\rceil = \lceil 1.5 \rceil = 2$$

THERE EXISTS A NEST WITH AT LEAST 2 PIGEONS

If we group everyone in this room by their day-of-the-month birthday, how many people will be in the largest group? (estimate & round as needed)

$$X = \frac{100}{31} = \lceil 3.2 \rceil = 4$$

Suppose all of New York City were to have a "hair-party" where they collect into groups of people who have exactly the same number of hairs on their head. Write one simple sentence which explains what the Pigeonhole principle tells us about this situation (google search, estimate & round as needed)

$$X = \left\lceil \frac{8 \cdot 10^6}{30 \cdot 10^3} \right\rceil = \left\lceil \frac{2}{30} \cdot 10^3 \right\rceil = .0\bar{6} \cdot 10^3 = .\bar{6} \cdot 10^2 = 66.\bar{6} \approx 67$$

(++) In a cocktail party with two or more people, is it possible that everyone has a different number of friends at the party? Assume that friendships are symmetric (if A is B's friend then B is A's friend).

Pigeonhole & counting Motivation:

$16^2 = 256$

Goal: publish everyone's grades publically online, each student's is associated with a "secret code"

- If you knew your code, you could identify your grade
- others don't know your code, they can't identify your grade

16^3

SECRET CODE	C1	D0	FF7	019	2AB	3CA	B4F
GRADE	A	A-	B-	B	A	A-	B+

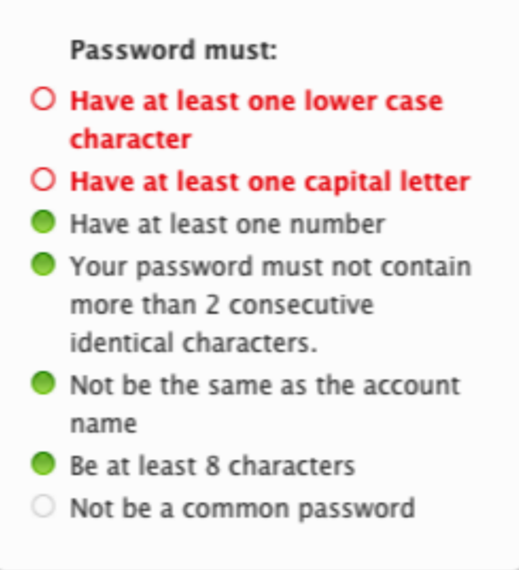
Suppose there are 800 students in the class and the secret code is a two-digit hex number. Are there enough secret codes for all students?

$$x = \left\lceil \frac{800}{256} \right\rceil = \lceil 3.\!\!\ldots \rceil = 4$$

Counting Motivation:

If a computer can guess 1000 times a second, how long does it take to guess a password which is:

- 4 lowercase characters? (a, b, c, d, ...) $\leftarrow \dfrac{x}{1000}$
- meets the requirements to the right



$$L = \{a, b, c, \dots, x, y, z\}$$

$$|L| = 26$$

$$|L \times L \times L \times L| = |L| \cdot |L| \cdot |L| \cdot |L|$$
$$= 26^4$$

NOTATION

aaab   baaa

SET

$\{a, b, c\}$

NO REPEATS

UNORDERED

$\{a,b\} = \{b,a\}$

TUPLE

(a, b, c, a)

MAY REPEAT

ORDER MATTERS

$(a,b) \neq (b,a)$

Set Operation: Cartesian Product

The cartesian product of A and B (A x B) is the set of all tuples, one item from A and the next from B

$A = \{1, 2\}$

$B = \{2, 3, 4\}$

$A \times B = \{(1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,3)\ (2,4)\}$

FIRST ITEM FROM A

SECOND ITEM FROM B

$A = \{1, 2\}$ $B = \{3, 4\}$ $(\_, \_)$

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

$(3,2) \notin A \times B$

is $A \times B = B \times A$ $(3,2) \in B \times A$

Set Operation: Cartesian Product (detail)

Example sets:

A = {1, 2}   B = {3, 4}   C = {5, 6}

Cartesian product of more than two sets:

$$A \times B \times C =$$

$$\{(1,3,5),\ (1,3,6),\ (1,4,5),\ (1,4,6),\ (2,3,5),\ (2,3,6),\ (2,4,5),\ (2,4,6)\}$$

The cartesian product is ordered

$$A \times B \neq B \times A$$

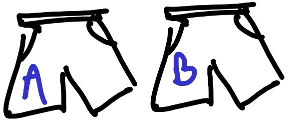
$$\{(1,3)(1,4)(2,3)(2,4)\} \qquad \{(3,1)(4,1)(3,2)(4,2)\}$$

GETTING DRESSED

ZZZ

My 3 year-old daughter has:

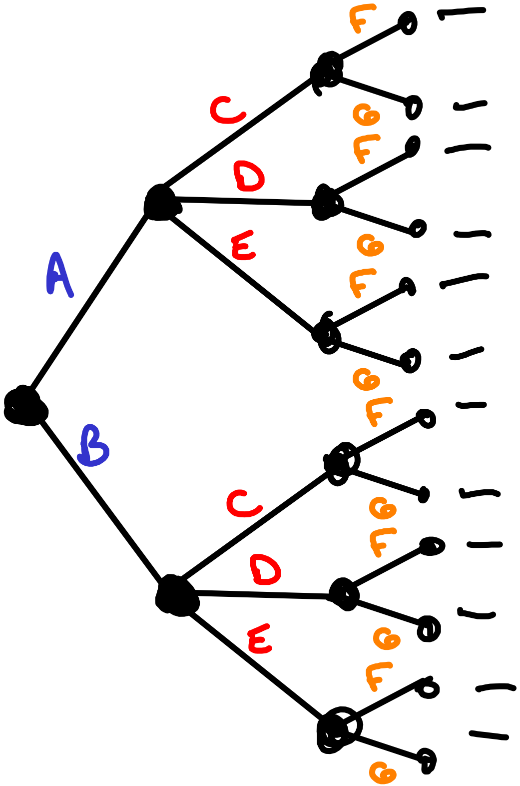
- 2 pants



- 3 shirts

- 2 pairs of socks

How many unique outfits can she wear?



12 OUTFITS

$= 2 \cdot 3 \cdot 2$

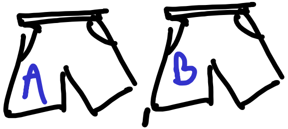
F, A, C

BUTT CHEST FEET

GETTING DRESSED

My 3 year-old daughter has:

- 2 pants

$\to P = \{$  $\}$

- 3 shirts

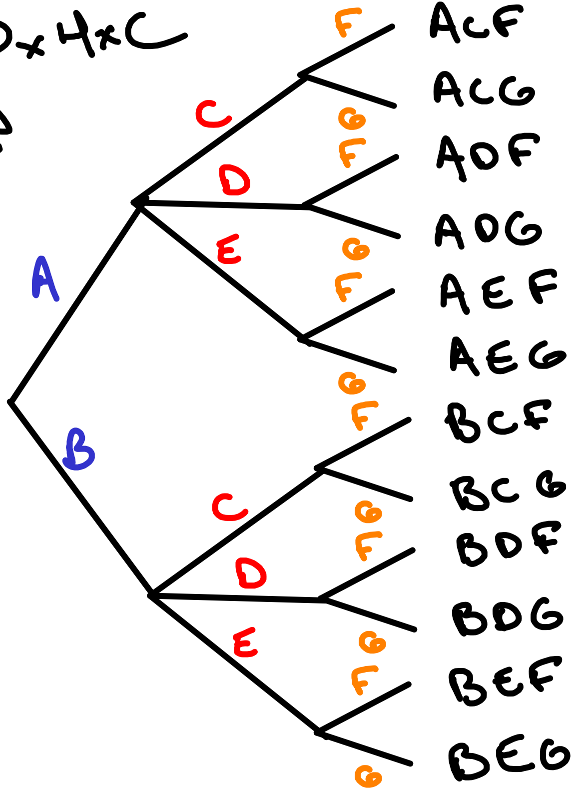
$\to H = \{$  $\}$

- 2 pairs of socks

$\to C = \{$  $\}$

How many unique outfits can she wear?

$P \times H \times C$



$|P \times H \times C|$

(A, D, F)

(A, D, G)

12 outfits

$= 2 \cdot 3 \cdot 2$

$= |P| \cdot |H| \cdot |C|$

$= |P \times H \times C|$

$X = \{\underline{a}, \underline{b}, c\}$ $\quad Y = \{\underline{a}, \underline{b}\}$

$$X \times Y = \{(a,a)\ (a,b)\ (b,a)\ (b,b)\ (c,a)\ (c,b)\}$$

Product Rule

(ξ, ξ)

The number of items in a cartesian-product is the product (multiplication) of items in each set:

$$|A \times B| = |A| \times |B|$$

CARTESIAN PRODUCT OF A AND B

MULTIPLICATION OF TWO NUMBERS

(ALWAYS USE "X" NOTATION)

MULTI-SET PRODUCTS TOO!

$$|A \times B \times C| = |A| \times |B| \times |C|$$

In Class Activity: Return of Password Counting

$L = \{a, b, c, \ldots, x, y, z\}$

How many passwords of length 4 can be made from lowercase letters?

$|L \times L \times L \times L| = |L| \cdot |L| \cdot |L| \cdot |L| = 26^4$

How many passwords of length 4 can be made from lower or upper case letters?

$|A \times A \times A \times A| = |A| \cdot |A| \cdot |A| \cdot |A| = 52^4$ $A = \{a, b, c, \ldots, x, y, z, A, B, C, \ldots, X, Y, Z\}$

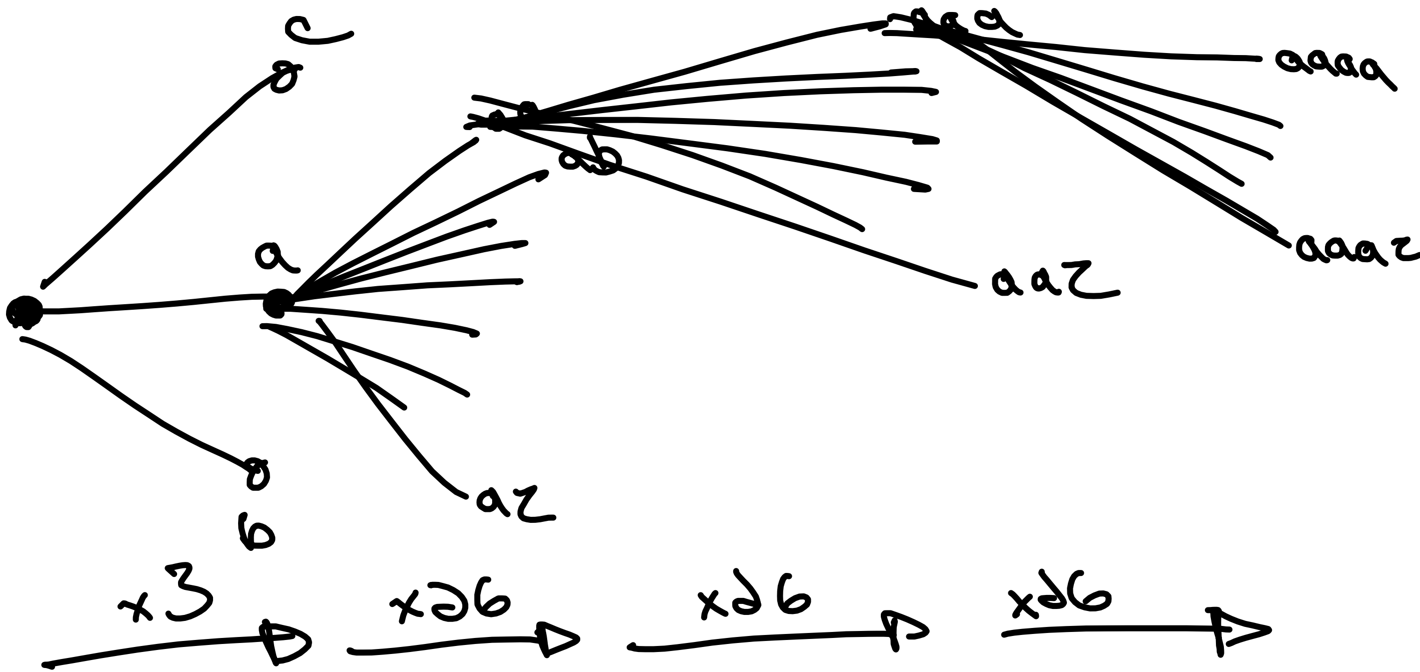
How many passwords of length 4 can be made from lowercase letters if the first letter must be 'a'?

26^3

$|\{a\} \times L \times L \times L| = |\{a\}| \cdot |L| \cdot |L| \cdot |L| = 26^3$

How many passwords of length 4 can be made from lowercase letters if the first letter must be 'a', 'b', or 'c'?

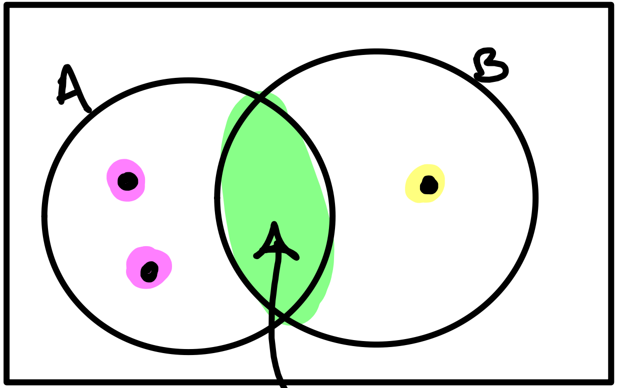
$|\{a, b, c\} \times L \times L \times L| = 3 \cdot 26^3$



c
a
b
az
ab
aaz
aaa
aaaa
aaaz
x3
x26
x26
x26

Sum Rule: counting unions of disjoint sets

If sets A and B are disjoint (no item is in both) then items in A union B is items in A plus items in B:



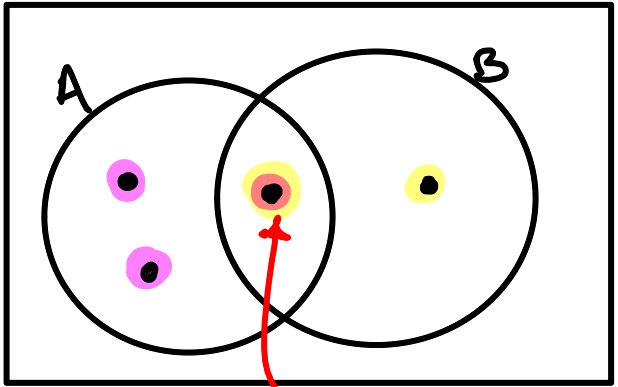
$$|A \cup B| = |A| + |B|$$

$$= 2 + 1 = 3$$

NOTICE: NO ITEM IS IN BOTH A AND B $\longleftrightarrow$ $A \cap B = \emptyset$

Sum Rule: won't work when sets share an item (i.e. not disjoint)

If sets A and B are disjoint (no item is in both) then items in A union B is items in A plus items in B:



$$|A \cup B| \neq |A| + |B|$$

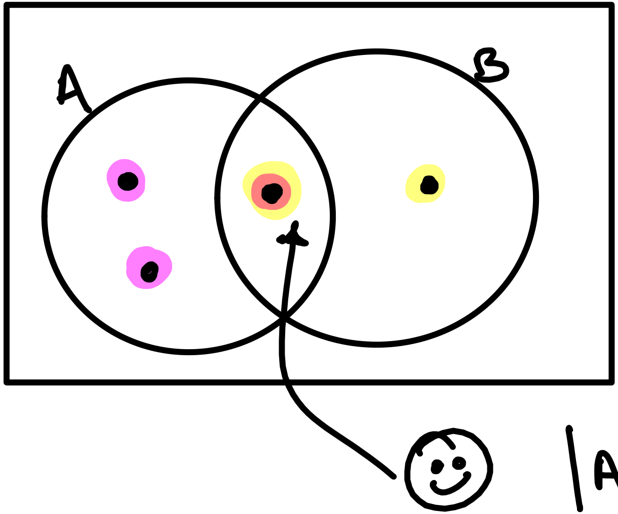
$$= 3 + 2 = 5$$

⚠ $|A| + |B|$ COUNTS ITEMS IN $|A \cap B|$ TWICE

Principle of Inclusion & Exclusion (PIE) (2 sets): Counting unions

For any sets A and B (maybe disjoint, maybe they share an item):

number of items in A union B = items in A + items in B - items in A intersect B



$$|A \cup B| = |A| + |B| - |A \cap B|$$

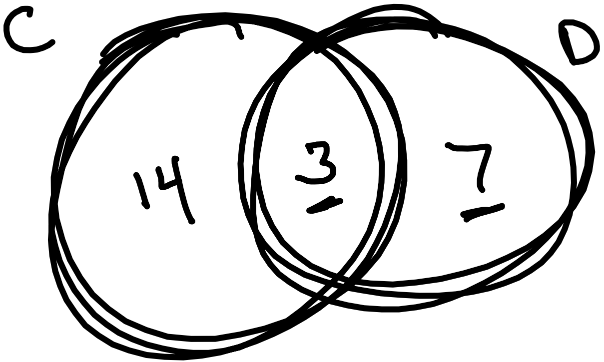
$$= 3 + 2 - 1 = 4$$

$|A| + |B|$ COUNTS ITEMS IN $|A \cap B|$ TWICE
SO WE CORRECT

$|C| = 17$

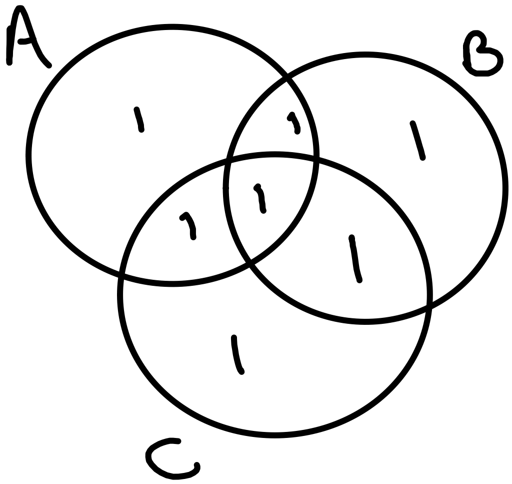
$|D| = 10$

$3 = |C \cap D|$



$$|C \cup D| = |C| + |D| - |C \cap D|$$
$$= 17 + 10 - 3 = 24$$

Principle of Inclusion & Exclusion (PIE) (3 sets): Counting unions which may or may not share an item

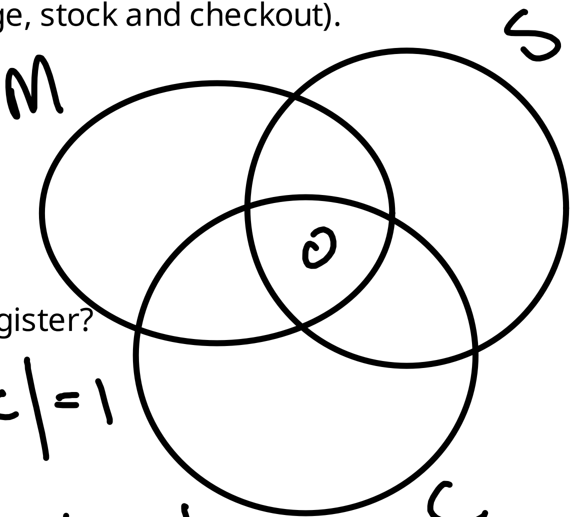


$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Practice together: 3 set PIE problem

A grocery store has 17 total employees who perform 3 roles (manage, stock and checkout).
(None of these 17 don't perform one of these 3 roles).
The following is a list of the training the 17 employees have.
- 3 are trained as managers $|M|=3$
- 10 are trained to stock groceries $|S|=10$
- 7 are trained to work the cash register $|C|=7$
- 1 employee has 'double-training' in every pair of jobs

How many employees are trained to manage, stock and work the register?



$$|M \cap S| = 1 \qquad |S \cap C| = 1 \qquad |M \cap C| = 1$$

C

$$|M \cup S \cup C| = |M| + |S| + |C| - |M \cap S| - |S \cap C| - |M \cap C| + |M \cap S \cap C|$$

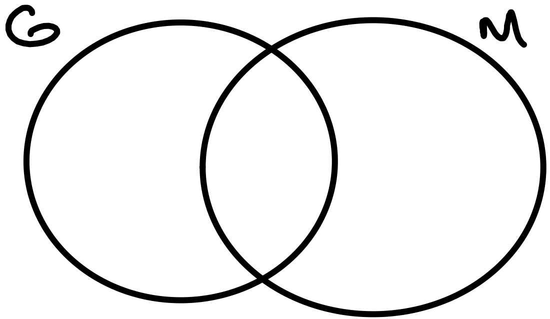
$$17 = 3 + 10 + 7 - 1 - 1 - 1 + |M \cap S \cap C|$$

In Class Assignment: 3 set PIE

Of the 196 kindergarden students who like either gym, music or art:

45 like gym class
90 like music class
100 like art class
20 like both gym and music
13 like both gym and art
7 like both art and music

- how many students like gym or music?
- how many students like all 3 subjects?
- how many students like gym but nothing else?



$$|G \cup M| = |G| + |M| - |G \cap M|$$
$$= 45 + 90 - 20$$
$$= 115$$

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7 like both art and music

- how many students like gym or music?
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- how many students like gym but nothing else?

$$|G \cup M \cup A| = |G| + |M| + |A| - |G \cap M| - |G \cap A| - |M \cap A| + |G \cap M \cap A|$$

$$196 = 45 + 90 + 100 - 20 - 13 - 7 + |G \cap M \cap A|$$

$$|G \cap M \cap A| = 1$$

In Class Assignment: 3 set PIE

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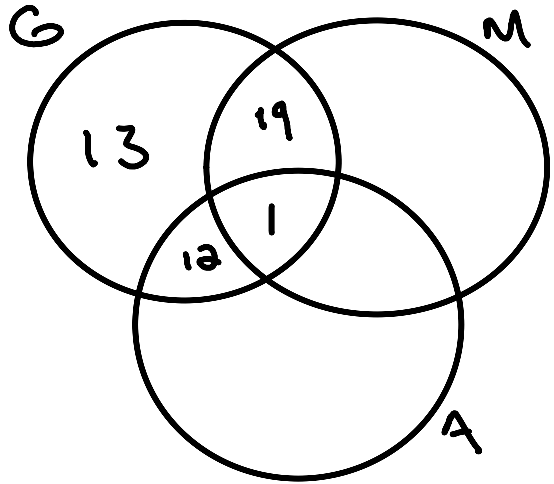
20 like both gym and music
13 like both gym and art
7 like both art and music

- how many students like gym or music?
- how many students like all 3 subjects?
- how many students like gym but nothing else?

$45 - 19 - 1 - 12$

$25 - 12$

13



$$|G \cup M \cup A| = |G| + |M| + |A| - |G \cap M| - |G \cap A| - |M \cap A| + |G \cap M \cap A|$$

In Class Assignment: 3 set PIE

Of the 196 kindergarden students who like either gym, music or art:

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45-

- how many students like gym or music?
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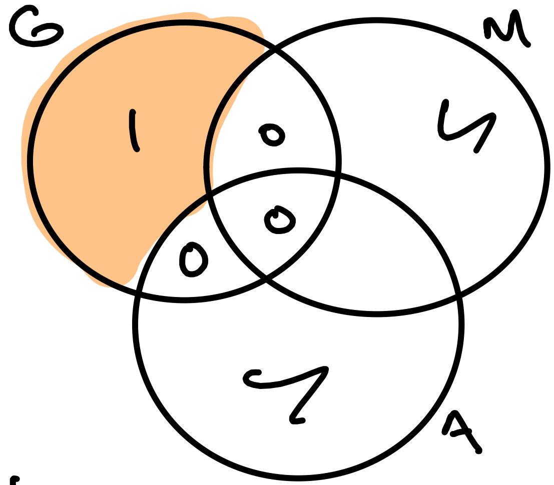
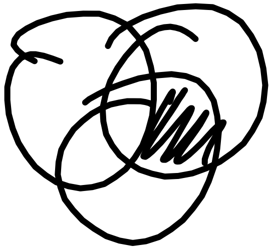


$$|G-M-A| = |G| - |G\cap M| - |G\cap A| + |G\cap M\cap A|$$

$8 \text{ MOD } 4 = x$

$8 = 4 \cdot 2 + \underline{0}$

PROBLEM !



$\underline{x} = (1, 5, 9, 13, \ldots)$

$\rightarrow x \in \{1, 5, 9, 13, \ldots\}$