Agenda

Professor Hamlin
Day 13

1. Admin
2. Joint Probability Dist.
3. Marginalization
4. Conditional Probability
5. Bayes Rule
6. Independence

Review

Probability - Experiment, outcomes, sample space, distribution, random variable, event

Expected Value - "average outcome"

$$E[X] = \sum_{x \in S} x \cdot Pr[X=x]$$

Variance: "how far from expected value"

$$Var(X) = E[X^2] - (E[X])^2$$

1. What is the probability of rolling ≤ 2 on the die w/ following #'s 1, 1, 1, 2, 6, 6

$$Pr[X \leq 2] = \frac{4}{6}$$

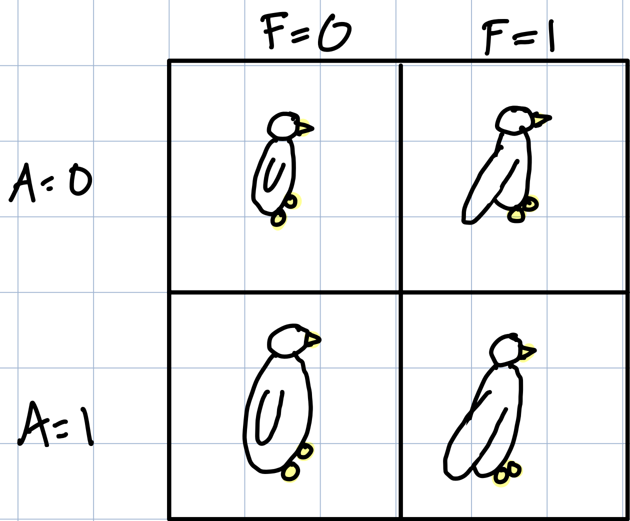
2. What is Expected Value of the die?

$$E[X] = 1 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{6} + 6 \cdot \tfrac{1}{3}$$

$$= \boxed{2 \tfrac{5}{6}}$$

Joint Probability : a dist. over more than 1 Random variable at a time

Let $A=1$ if penguin is adult (0 otherwise)
Let $F=1$ if penguin has big flippers (0 otherwise)



	F=0	F=1
A=0	3/12	2/12
A=1	1/12	6/12

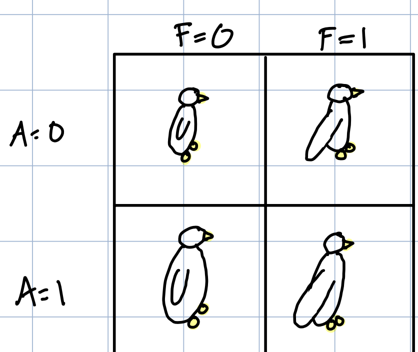
everything has to sum to 1!

half of penguins are adults w/ big flipper

happening at same time

$\Pr[A=0, F=1]$ is how we express it w/ math notation

Marginalization : removing a random variable from probability dist. (e.g. what fraction of penguins are adults)



	F=0	F=1
A=0	3/12	2/12
A=1	1/12	6/12

This row is all the adult penguins!

$$\Pr[A=1, F=0] + \Pr[A=1, F=1]$$
$$\frac{1}{12} + \frac{6}{12} = \boxed{7/12}$$

To compute $\Pr[A=a]$ sum up $\Pr[A=a, B=?]$ for all outcomes in sample space of B

$$\Pr[A=a] = \sum_{b \in S} \Pr[A=a, B=b]$$

Exercise

C = color of penguin (red, blue, green)
A = penguin is adult (1) or 0 otherwise

	C = red	blue	green
A = 0	1/12	3/12	0/12
A = 1	2/12	1/12	5/12

1) $\Pr[C = \text{blue}]$

$\frac{3}{12} + \frac{1}{12} = 1/3$

2) $P[C=\text{red}] + \Pr[C=\text{green}]$

$\frac{1}{12} + \frac{2}{12} + \frac{0}{12} + \frac{5}{12} = \frac{8}{12}$

3) $\Pr[A=1]$

$\frac{2}{12} + \frac{1}{12} + \frac{5}{12} = \boxed{\frac{8}{12}}$

$1 - \Pr[C = \text{blue}]$

Conditional Probability: "if x then the probability of y is?"

$C = 1$ indicate if person has covid

$T = 1$ indicates test is positive

1) What is prob person has positive test

$Pr[T=1]$

2) If person has covid then what is prob of positive test?

$Pr[T=1 \mid C=1]$ → This is read as "Given $C=1$"

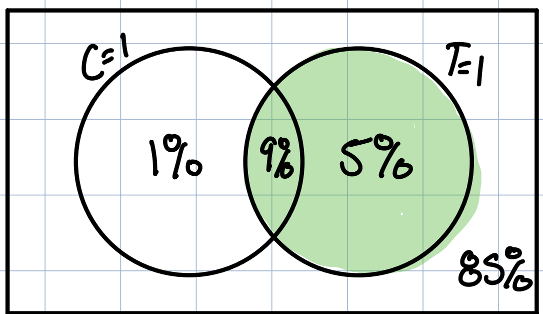
3) Prob person has covid given positive test?

"if pos test then prob covid?"

$Pr[C=1 \mid T=1]$

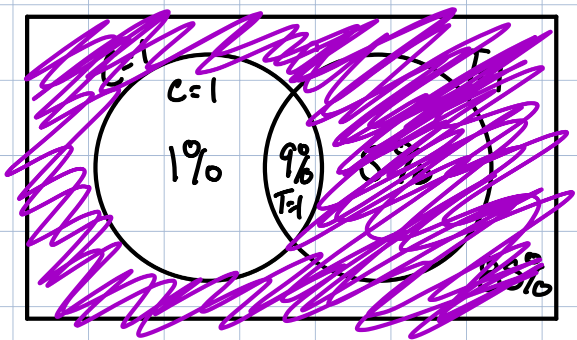
So how do we calculate it? Let's talk intiution

1) What is prob person has positive test



$Pr[T=1] = 9\% + 5\% = 14\%$

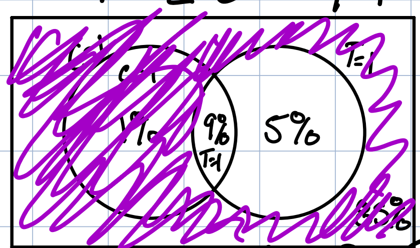
2) If person has covid then what is prob of positive test? $\Pr[T=1 \mid C=1]$



What fraction of our new world T=1

$$\frac{9\%}{1\% + 9\%} = \boxed{90\%}$$

3) Prob person has covid given positive test?
$\Pr[C=1 \mid T=1]$



$$\frac{9\%}{9\% + 5\%} = \boxed{\frac{9}{14}}$$

Conditional $\Pr[X=x \mid Y=y]$ is prob of $\Pr[X=x]$ when we constrain ourselves to the world of $Y=y$

Formally:

Prob x & y happen together

$$\Pr[X=x \mid Y=y] = \frac{\Pr[X=x, Y=y]}{\Pr[Y=y]}$$

Prob x happens given y happening

Prob y happens

Exercise! S: twitter sentiment score ($1 =$ good, $0 =$ neutral, $-1 =$ bad)
B: Bitcoin price ($1 =$ up, $-1 =$ down)

	$S=-1$	$S=0$	$S=1$
$B=-1$	19%	27%	5%
$B=1$	8%	21%	20%

1) Compute $\Pr[S=-1 \mid B=1]$ and $\Pr[S=-1]$ and explain # in english.

$\Pr[S=-1] = 19 + 8 = \boxed{27\%}$

$$\Pr[S=-1 \mid B=1] = \frac{\Pr[S=-1,\ B=1]}{\Pr[B=1]}$$

$$= \frac{.08}{.08 + .21 + .2} = \boxed{16\%}$$

Given Bitcoin prices rising negative sentiment is less likely

2) Compute $\Pr[B=1 \mid S=-1]$ and $\Pr[B=1]$ and explain # in english.

$\Pr[B=1] = .08 + .21 + .2 = 49\%$

$$\Pr[B=1 \mid S=-1] = \frac{\Pr[S=-1,\ B=1]}{\Pr[S=-1]}$$

$$= \frac{8\%}{27\%} \approx 30\%$$

Given negative sentiment it is less likely for bitcoin prices to rise.

Note we can manipulate the conditional prob. expression

$$Pr[A=a \mid B=b] = \frac{Pr[A=a, B=b]}{Pr[B=b]}$$

~or~

$$Pr[A=a, B=b] = Pr[A=a \mid B=b] \cdot Pr[B=b]$$

Multiplying conditional prob w/ the probability of condition yields prob both outcomes happen together

Bayes Rule: if given $Pr[A=a \mid B=b]$ how to get $Pr[B=b \mid A=a]$?

$$Pr[A=a \mid B=b] \cdot Pr[B=b] = Pr[A=a, B=b]$$

and

$$Pr[B=b \mid A=a] \cdot Pr[A=a] = Pr[A=a, B=b]$$

$$\Rightarrow Pr[A=a \mid B=b] \cdot Pr[B=b] = Pr[B=b \mid A=a] \cdot Pr[A=a]$$

$$\Rightarrow \boxed{Pr[A=a \mid B=b] = \frac{Pr[B=b \mid A=a] \cdot Pr[A=a]}{Pr[B=b]}}$$

Bayes Rule

e.g. if given variables in one order find them in another

Helpful Note:

marginalization

$$Pr[B=b] = \sum_{a \in S} Pr[A=a, B=b]$$

definition conditional prob

$$= \sum_{a \in S} Pr[B=b \mid A=a] \cdot Pr[A=a]$$

Why is this helpful?

$$Pr[A=a \mid B=b] = \frac{Pr[B=b \mid A=a] \cdot Pr[A=a]}{Pr[B=b]}$$

we now can calculate this!

$$Pr[A=a \mid B=b] = \frac{Pr[B=b \mid A=a] \cdot Pr[A=a]}{\sum_{x \in S} Pr[B=b \mid A=x] \cdot Pr[A=x]}$$

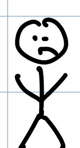
Another form of Bayes Rule

Example Given flu occurs in 4% of population, what is the prob one has flu given they test positive?

$F=0$  (healthy) $T=0$ $\ominus$ negative

$Pr[T=0 \mid F=0] = .9$

$Pr[T=0 \mid F=1] = .01$

$F=1$  Flu $T=1$ $\oplus$ positive

$Pr[T=1 \mid F=0] = .1$

$Pr[T=1 \mid F=1] = .99$

$Pr[F=1] = .04 \Rightarrow Pr[F=0] = .96$

Asking $Pr[F=1 \mid T=1] = ?$

$$Pr[F=1 \mid T=1] = \frac{Pr[T=1 \mid F=1] \cdot Pr[F=1]}{Pr[T=1]}$$

$$Pr[T=1] = Pr[T=1 \mid F=0] \cdot Pr[F=0] + Pr[T=1 \mid F=1] \cdot Pr[F=1]$$

$$= .1 * .96 + .99 * .04$$

$$= .1356$$

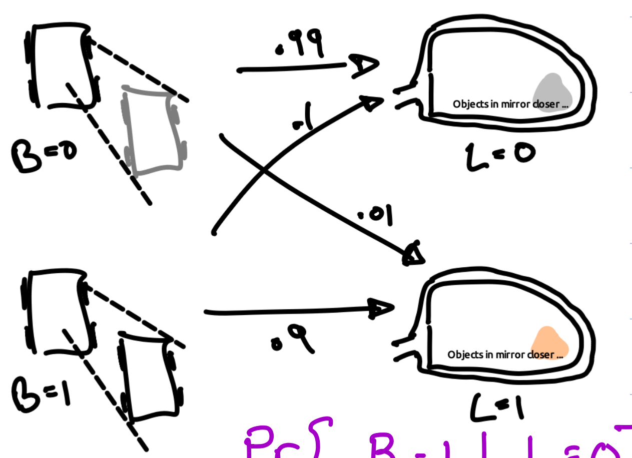
$$\boxed{Pr[T=1, F=0] + Pr[T=1, F=1]}$$

$$\frac{.99 \cdot .04}{.1356} \approx 29\%$$

Exercise

A blind spot monitor produces a warning light (L=1) when it estimates that a car is in one's blind spot (B=1). Given that the light is off, whats the probability that a car is one's blind spot? (Assume that a car is in your blindspot 2 percent of the time while driving.)

$Pr[B=1] = 2\%$



$Pr[L=0 \mid B=0] = .99$

$Pr[L=0 \mid B=1] = .1$

$Pr[L=1 \mid B=0] = .01$

$Pr[L=1 \mid B=1] = .9$

$$Pr[B=1 \mid L=0] = \frac{Pr[L=0 \mid B=1] \cdot Pr[B=1]}{Pr[L=0]}$$

$$Pr[L=0] = Pr[L=0 \mid B=0] \cdot Pr[B=0] + Pr[L=0 \mid B=1] \cdot Pr[B=1]$$

$$= .99 \cdot .98 + .1 \cdot .02$$

$$\frac{.1 \cdot .02}{.99 \cdot .98 + .1 \cdot .02} \approx .00205$$

Independence

e.g dice, coin flip

Intuition: if random variables X & Y are independent if observing any outcome of one does not impact the outcome of the other

Math: $Pr[X=x, Y=y] = Pr[X=x] \cdot Pr[Y=y]$

~and~

$Pr[X=x \mid Y=y] = P[X=x]$

Example

$Pr[X=1] = 1/4$, 0 otherwise

$Pr[Y=1] = 1/10$, 0 otherwise

	$x=0$	$x=1$
$y=0$	$\frac{3}{4} \cdot \frac{9}{10}$	$\frac{1}{4} \cdot \frac{9}{10}$
$y=1$	$\frac{3}{4} \cdot \frac{1}{10}$	$\frac{1}{4} \cdot \frac{1}{10}$

Exercise) Two unfair coins $Pr[C1=H] = 3/4$

$Pr[C2=H] = 2/3$

What is the probability of getting...

... $Pr[C1=H, C2=T] = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$

... $Pr[C1=T, C2=T] = \frac{1}{4} \cdot \frac{1}{3} = 1/12$