CS1800 day 3

Admin:
- hw1 released today (due the following friday, as nearly all HWs are)
- tutoring groups

Content:
- Two's complement (system to represent negative binary numbers)
- Overflow
- Floating point (system to represent non-whole numbers) (if time)

Whats the difference between operating in base-b and operating in base-b on a computer?

Computers store all values with the same number of bits

why? quicker / easier

Assume: a computer is using a 3-bit representation of values. How does it compute & store the following?

$$(111)_2 + (001)_2 = (1000)_2$$

$$7 + 1 = 8$$

ONLY THESE THREE BITS CAN BE STORED

THIS IS DISCARDED

For today: assume we're working with values on a computer

- all values are N-digits
  (you'll be given this info in problem statement)

- discard the most significant (left-most) digits if needed
  (as shown in green on last slide)

LARGEST PLACE VALUE

MAGNITUDE

Number Systems:

Currently we're missing:

- negative values (e.g. -43)
- non-whole values (e.g. 321.12358)

Number systems:

- Unsigned Integers:

  can represent whole, non-negative numbers

  everything we've done so far are unsigned integers (we just didn't cover name until now)

  e.g. (110)_2 = 6

- Two's Complement:

  can represent whole (potentialy negative) numbers

  (will study today)

- Floating Point Values:

  non whole-numbers

  (will study today if time)

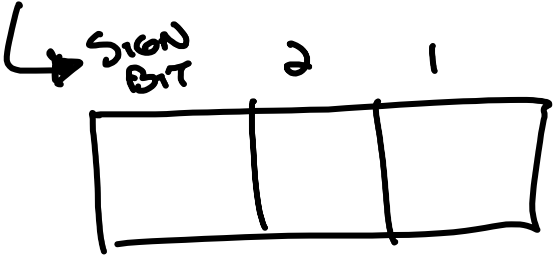
Sign bit*:

A not-so-great number system for negative values

3 BIT "SIGN BIT"

1 IF NEGATIVE

0 OTHERWISE



$(001)_2 = 1$

$(000)_2 = 0$

$(111)_2 = -3$

$(101)_2 = -1$

Sign Bit: Problems

$$\begin{array}{r} 111 \\ +\ 001 \\ \hline \cancel{1}000 \end{array}$$

No unique zero

$$(000)_2 = 0$$

$$(100)_2 = -0$$

Our operations yield incorrect results

Discard

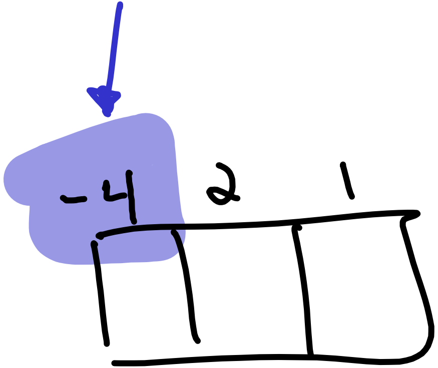
$$(111)_2 + (001)_2 = (\cancel{1}000)_2$$

$$-3 + 1 = 0$$

N	2	1
1	1	0

Two's complement: A better way to store negative numbers

Big idea: the most significant (biggest) place value is negative, all others are positive



Example: 3-bit two's complement

-4 2 1	
100	-4
101	-3
110	-2
111	-1
000	0
001	1
010	2
011	3

3+3

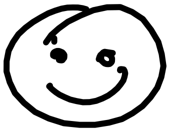
$$\begin{array}{r} {\scriptstyle 11} \\ 011 \\ +\ 011 \\ \hline 110 \end{array}$$

-2

Two's complement: Problems solved

Unique zero

$$(000)_2 = 0$$

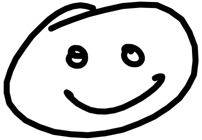


Our operations yield correct* results

Discard

$$(111)_2 + (001)_2 = (\cancel{1}000)_2$$

$$-1 + 1 = 0$$



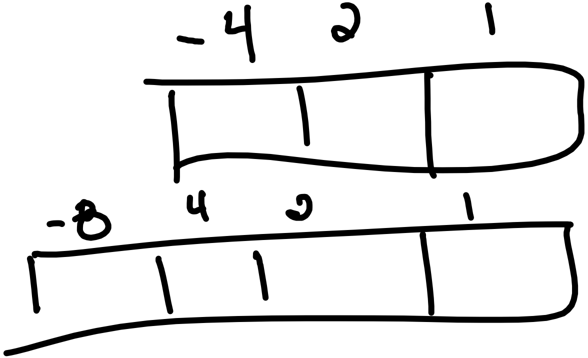
* Assumes that correct result may be represented (more later)

In Class Activity:

If possible, convert each of the following values to the given number system. If not possible, justify why.

(Use guess-and-check as needed, a reliable decimal-to-2's-complement method coming shortly)

0	unsigned 2 bit integer
-2	unsigned 3 bit integer
0	3 bit 2's complement
-4	3 bit 2's complement
-4	4 bit 2's complement
5	4 bit 2's complement
10	4 bit 2's complement
-3	4 bit 2's complement



(++) What does the 2's complement idea look like in a base which isn't 2? Does it also have the properties we love so much in binary (unique zero, addition operations still work)?

0 unsigned 2 bit integer

$(00)_2$

2	1

-2 unsigned 3 bit integer: impossible!

- unsigned number systems only represent non-negative values
- each place value is positive, we may not represent a negative value

0 3 bit 2's complement

$(000)_2$

-4	2	1
1	0	0

-4 3 bit 2's complement

$(100)_2$

-4 4 bit 2's complement

$(1100)_2$

-8	4	2	1
1	1	0	0

$-8+4+0+0=-4$

5 4 bit 2's complement

$(0101)_2$

8	4	2	1
0	1	0	1

10 4 bit 2's complement

-8	4	2	1
	1	1	1

$4+2+1=7$

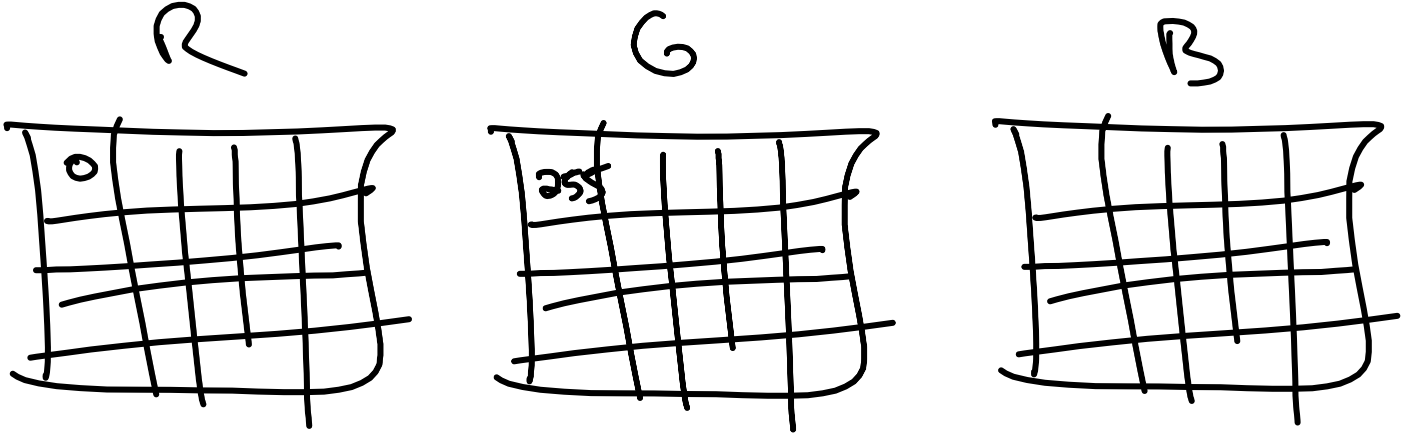
10 is impossible! biggest value in
4 bit 2's comp is (0111)_2 = 4 + 2 + 1 = 7

-3 4 bit 2's complement

$-8+X=-3$

$x=5$

-8	4	2	1
1	1	0	1


R
G
B
0
255

What values can we represent with N bits?

Unsigned Integers

2^{N-1}	. . .		2^1	2^0
1	1	1	1	1

SMALLEST VALUE

$$0$$

LARGEST VALUE

$$2^N - 1$$

Two's Complement

-2^{N-1}	. . .		2^1	2^0
0	1	1	1	1

SMALLEST VALUE

$$-2^{N-1}$$

LARGEST VALUE

$$2^{N-1} - 1$$

16	8	4	2	1
1	1	1	1	1
4	3	2	1	0

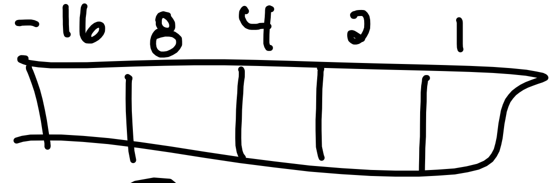
$$1 + (x)_{10} = (11111)_2 + (1)_2$$

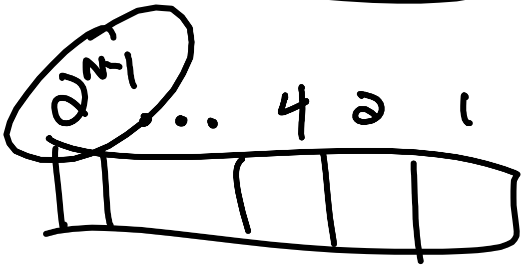
$$= (100000)_2$$

$$= 2^5$$

$$\begin{array}{r} \scriptstyle 1\,1\,1\,1\, \\ 11111 \\ +\ 00001 \\ \hline 100000 \end{array}$$

$$x = 2^5 - 1$$



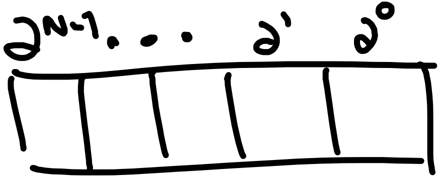


$N = 5$ $\quad -2^{5-1} = -16$

16	8	4	2	1
0	0	0	0	0

What values can we represent with N bits? (representability)

Unsigned Integers



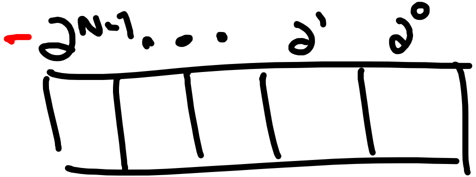
SMALLEST VALUE

0

LARGEST VALUE

$2^N - 1$

Two's Complement



SMALLEST VALUE

-2^{N-1}

LARGEST VALUE

$2^{N-1} - 1$

We can represent all whole values from smallest to largest (including smallest & largest)
(we won't justify this)

OVERFLOW

$$\begin{array}{ccc} -4 & 2 & 1 \\ \boxed{1} & \boxed{1} & \boxed{1} \end{array} = -4+2+1 = -1 \qquad \begin{array}{ccc} -4 & 2 & 1 \\ \boxed{0} & \boxed{0} & \boxed{1} \end{array} = 1$$

Overflow: the outcome of an operation can't be represented in the given number system

example from earlier in lesson: $(111)_2 + (001)_2 = (\cancel{1}000)_2$

DISCARD

7 + 1 = 8 as 3 bit values $\qquad -1 + 1 = 0$

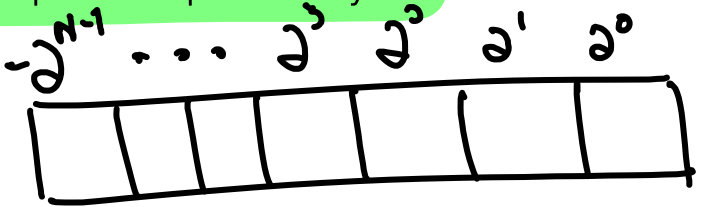
overflow since 8 can't be represented as a 3-bit value

Common misconception:

There are times when we discard a bit but result is correct (no overflow occurs)

punchline: bit discard not relevant when determining overflow

Decimal to N-bit two's complement: preliminary



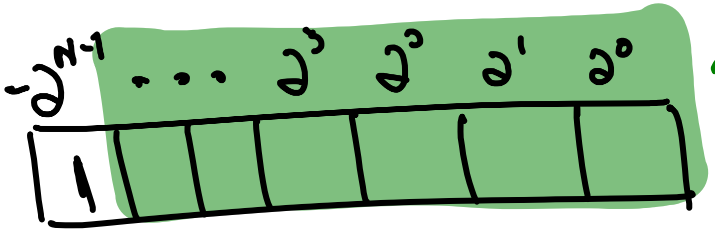
1. Validate that value can be represented as N-bit two's complement (see "representability")

2. If value is positive, its the same as N bit unsigned integer
   methods:
   - subtract largest power of two
   - Euclid's Division Algorithm

(if a non-negative number can be represented in n-bit 2's complement, its the same as n-bit unsigned)

3. If value is negative: see "x" method on next slide

Decimal to N-bit two's complement: "x" method for negative representable values



DEFINE X AS VALUE OF LAST N-1 BITS (UNSIGNED)

A. Solve for X
B. Represent X as N-1 bit unsigned int
C. Append a leading 1 to indicate the -2^{N-1}

$$\text{VALUE} = -2^{N-1} + X$$

SO THAT

$$X = \text{VALUE} + 2^{N-1}$$

"X" method example

Represent -5 in 4 bit 2's complement

$-2^{N-1} = -2^3 = -8$ SMALLEST

$2^{N-1} - 1 = 2^3 - 1 = 7$ LARGEST

-8	4	2	1
1	0	1	1

$-8 + x = -5$

$x = 3$

In Class Activity 2

SMALLEST = -32
LARGEST = 31

If possible, express each of the following as a 6 bit two's complement value. Use the "x" method where possible.

-5 $(111011)_2$
5
~~32~~

IMPOSSIBLE

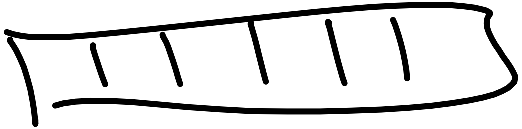
32 BIGGER THAN BIGGEST REPRESENTABLE

-32	16	8	4	2	1
1	1	1	0	1	1

$$-32 + x = -5$$

$$x = 27$$
$$= 16 + 11$$
$$= 16 + 8 + 3$$
$$= 16 + 8 + 2 + 1$$

5 IN 6-BIT 2's COMP



$(000101)_2$

(floating point if time)

Floating Point: Representing non-whole values

To express 12.345, rewrite it as:

$$12.345 = \underbrace{12345}_{\text{significand}} \times \underbrace{10}_{\text{base}}^{\overbrace{-3}^{\text{exponent}}}$$

ALSO KNOWN AS MANTISSA

big idea: the signifcand and exponent will always be whole values and we can store those!

A few notes about the "base"
- isn't the same base the number system for significand & exponent number system
  - (you can use base 10, as shown, and still store significand & exponent in binary)
- no need to store floating point base per individual value

IF TIME: NUMPY DOCUMENTATION

img credit: wikipedia