# CS1800 Day 9

Admin:

- exam1 is on Oct 15th
- hw3 due today, hw4 released today
- hw4 deadlines are funny (for exam):
    - includes content from day10 (next class)
    - solutions for hw4 released sunday oct 13 @ 12:01am, first thing in the morning
        - good news: allows you study
        - bad news: you may only use up to 1 late day on hw4
- Fri Oct 11:  "practice" exam together in class and work through any tough examples

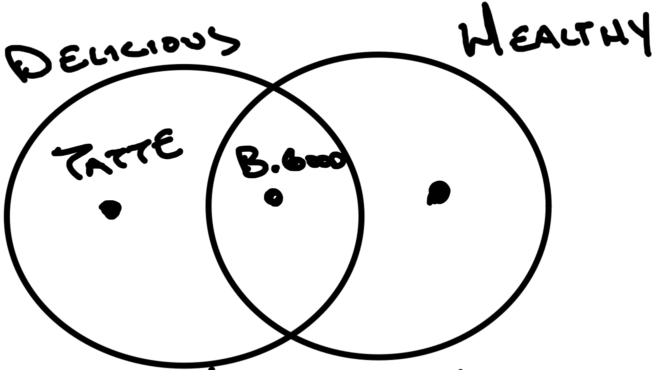
Content:

- review PIE & product rule
- permutations
- count by partition
- count by complement
- count by simplification

## Reminder: Sum Rule & Principle of Inclusion-Exclusion (Sum rule is a special case of PIE)

When counting the ways we can select one item from either of two sets:

How many different places could I go to for lunch at Northeastern?




$$|D \cup H| = |D| + |H| - |D \cap H|$$
$$= 2 + 2 - 1 = 3$$

Reminder: Product Rule (counting Cartesian Products)

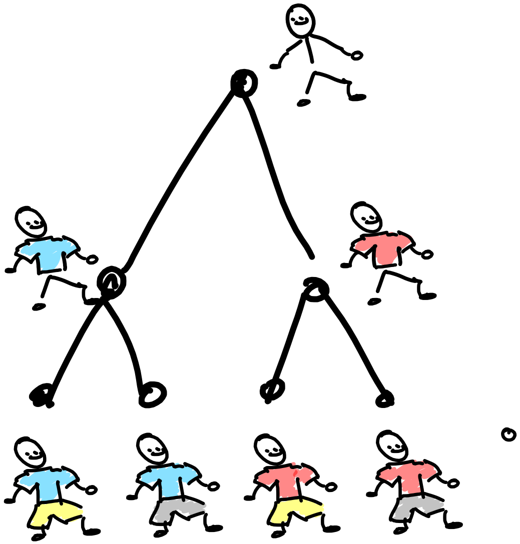
When counting the ways we can select an item from one set AND an item from another set:

How many different outfits can be formed?

$$|S \times P| = |S| \times |P| = 2 \cdot 2 = 4$$

$$S = \{\ ,\ \}$$

$$P = \{\ \ \}$$



# NOTATION (REMINDER)

**SET**

$\{a, b, c\}$

NO REPEATS

UNORDERED

**TUPLE**

$(a, b, c, a)$

MAY REPEAT

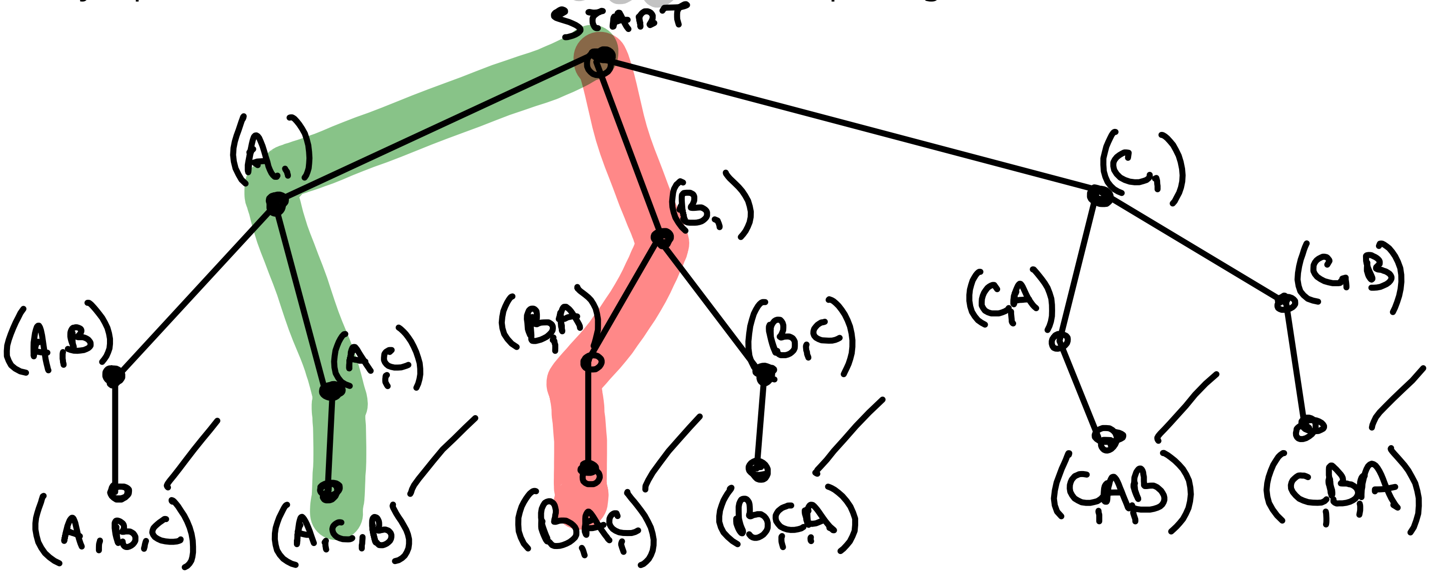
ORDER MATTERS

$(a,b) \neq (b,a)$

## Permutations: Travelling Salesperson


How many ways can a salesman order 3 city visits?

(How many tuples can we make from items A, B, C without repeating?



## Permutations: Travelling Salesperson

How many ways can a salesman order 3 city visits?

$P(3,3)$

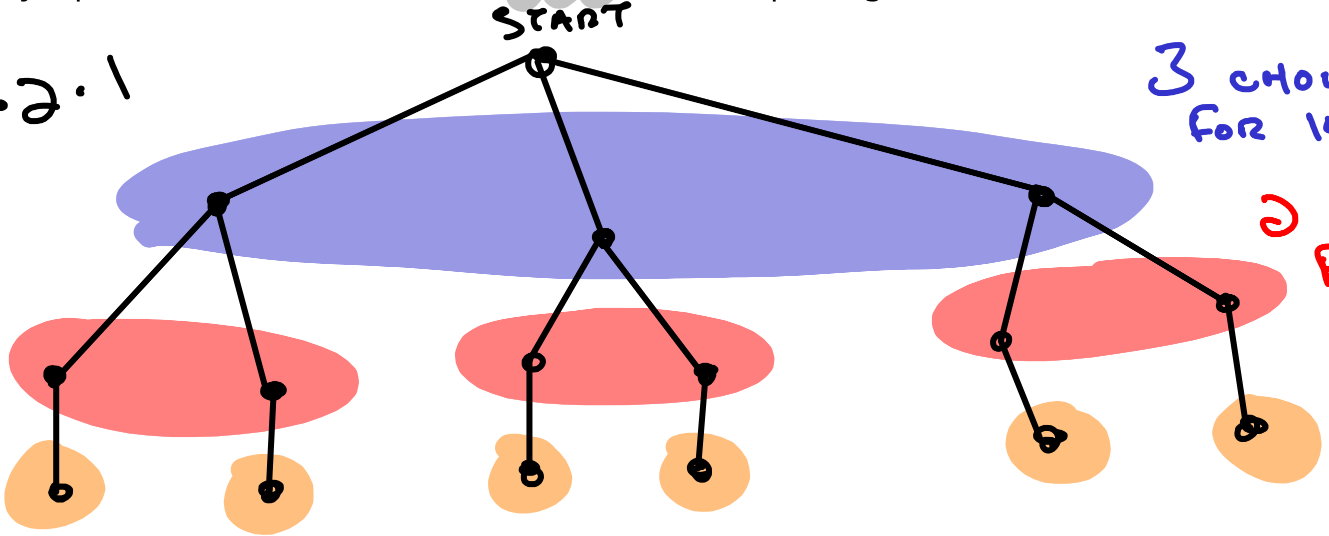
ATLANTA

CHICAGO

BOSTON

(How many tuples can we make from items A, B, C without repeating?)

$3 \cdot 2 \cdot 1$




3 CHOICES FOR 1ST CITY

2 CHOICES FOR 2ND

1 CHOICE FOR LAST CITY

Factorial:

"8 FACTORIAL"

$$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

CONVENTION $0! = 1$

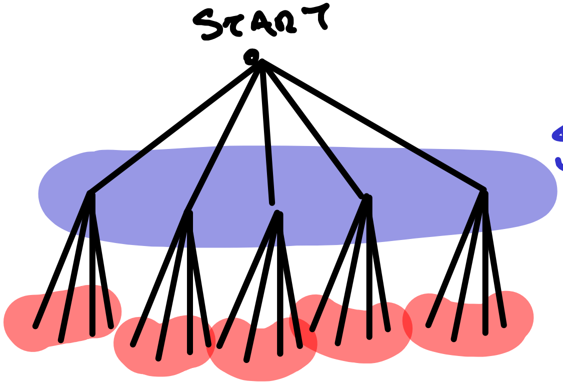
OUR SALESMAN (OF PREVIOUS SLIDE) HAD

$3! = 3 \cdot 2 \cdot 1$ TOTAL ORDERINGS OF 3 CITIES

## Permutations: A Travelling (lazy) Salesperson

How many ways can a salesman order 2 of 5 cities?

(How many tuples of length 2 can be made from A, B, C, D, E where no repeats allowed)?




5 CHOICES FOR 1ST CITY

4 CHOICES FOR 2ND CITY

$$\underline{5}\cdot\underline{4} = \frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1} = \frac{5!}{3!}$$

$$= \frac{5!}{(5-2)!}$$

## Permutations:

The number of ways of ordering k objects, from n total available is:

$$P(n,k) = \frac{n!}{(n-k)!}$$

PREVIOUS EXAMPLES

VISIT 3 OF 3 CITIES

$$P(3,3) = \frac{3!}{(3-3)!} = 6$$

VISIT 2 OF 5 CITIES

$$P(5,2) = \frac{5!}{(5-2)!} = 20$$

## In Class Activity

How many ways are there to order 5 people for a family portrait?

How many ways are there to order 6 of 20 people for a family portrait?

(If time): Plug a few of these factorials into calculator or google, how big of a factorial do you need to plug in until you "break" your computer?

In Class Activity

PEOPLE IN FAMILY = $\{A, B, C, D, E\}$

How many ways are there to order 5 people for a family portrait?

C B D E A

B C D E A

$$5! = P(5,5) = \frac{5!}{(5-5)!}$$

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

5

## In Class Activity

How many ways are there to order 6 of 20 people for a family portrait?

$$\underset{20}{\underline{\quad}} \quad \underset{19}{\underline{\quad}} \quad \underset{18}{\underline{\quad}} \quad \underset{17}{\underline{\quad}} \quad \underset{16}{\underline{\quad}} \quad \underset{15}{\underline{\quad}}$$

$$20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 = \frac{20!}{14!} = \frac{20!}{(20-6)!} = \underline{P(20,6)}$$

Factorials grow really quickly: (more on this when we study "function growth" later)

$10! \simeq 3.6 \cdot 10^{6}$ 3.5 MILLION

$20! \simeq 2 \cdot 10^{18}$ 2 MILLION BILLIONS

$19! \simeq 10^{17}$ SECONDS SINCE BIG BANG

$50! \simeq 10^{80}$ ATOMS IN UNIVERSE

$70! \simeq 10^{100}$ GOOGLE

Convention (in this class):

FEEL FREE TO LEAVE EXPRESSION AS

$P(5,3)$ OR $\dfrac{5!}{2!}$

Advice (for your future work):

Use scientific notation to get a sense of numbers

(challenging to get a sense of scale in P(50, 40), see previous slide)

Counting "moves":

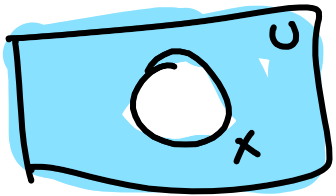
- count by complement
- count by partition
- count by simplify

## Count-by-complement

How many ways are there to order 5 people such that person A is not last?

$|U| = 5!$ WAYS OF ORDERING 5 PEOPLE

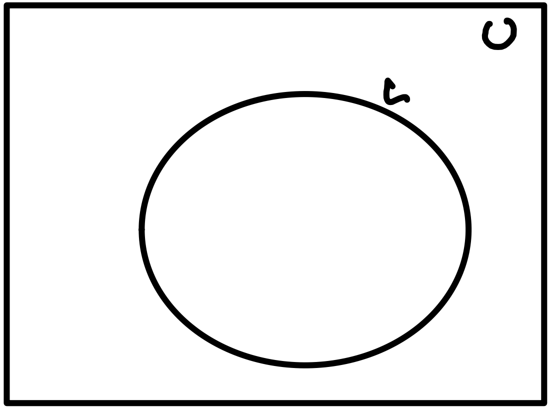
$\_\ \_\ \_\ \_\ \underline{A}$ $\quad |X| = 4!$ WAYS OF ORDERING 5 PEOPLE PERSON A IS LAST




$$5! - 4!$$

## Count-by-complement

How many ways are there to order 5 people such that person A is not last?



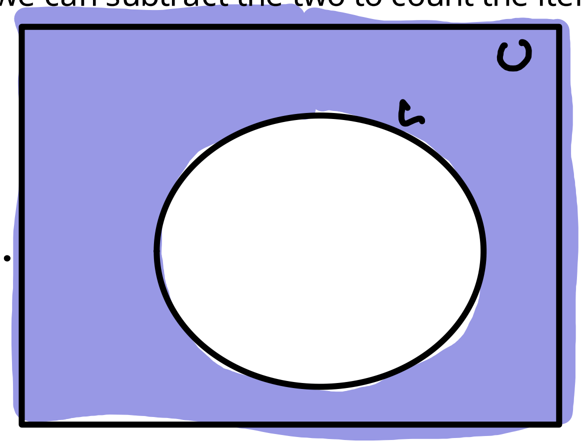

L = SET OF ORDERINGS WHERE A IS LAST

U = ALL ORDERINGS OF 5 PEOPLE

## Count-by-complement

How many ways are there to order 5 people such that person A is not last?

General approach: If we can count everything (U) and all items we're not interested in (L) then we can subtract the two to count the items of interest.




$$|U - L| = |U| - |L|$$

⚠ $|A - B| = |A| - |B|$ NOT TRUE IN GENERAL

Counting "moves":

- count by complement ✓
- count by partition
- count by simplify

## Count-by-partition: motivating example

$P \to$

$\underline{a}\ \underline{a}\ \underline{a}\ \underline{a}\ \underline{a}$

26 26 26 26 26

How many passwords can be made of lowercase letters which are no longer than 5 characters?

abcde

abc

① $P = P_5 \cup P_4 \cup P_3 \cup P_2 \cup P_1 \cup P_0$

PASSWORDS OF LENGTH 5

$|P| = 26^5 + 26^4 + 26^3 + 26^2 + 26^1 + 26^0$

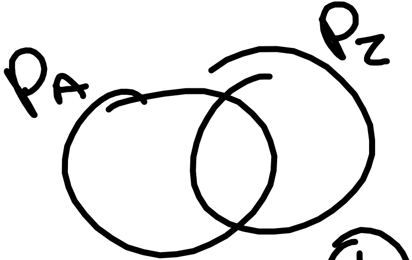
PASSWORDS START WITH A OR END WITH Z

$$|P| = |P_A| + |P_Z|$$

PASSWORDS START W/ A

PASSWORDS END W/ Z

A _ _ _ _ Z




① $P = P_A \cup P_Z$

② $P_A \cap P_Z \neq \emptyset$

## Partition: definition

Intuition: partition of set A divides its items into subsets where each item in A is in exactly one subset A_i

$A = \{1, 2, 3, 4\}$  $A_1 = \{2, 3\}, A_2 = \{1\}, A_3 = \{4\}$

A is partitioned by $A_1$ $A_2$ $A_3$

Definition: partition of set A is a set of subsets A_i where every item in A is in exactly one subset

every item in A is in, at most, one subset A_i

subsets are pairwise-disjoint:

intersection of every pair of subsets is empty

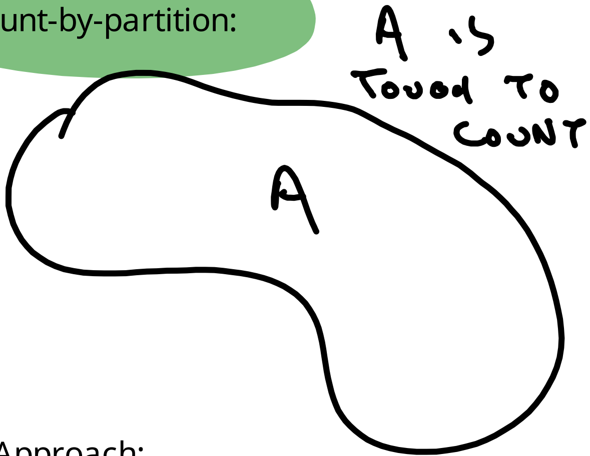
$\forall i, j \quad i \neq j \rightarrow A_i \cap A_j = \emptyset$

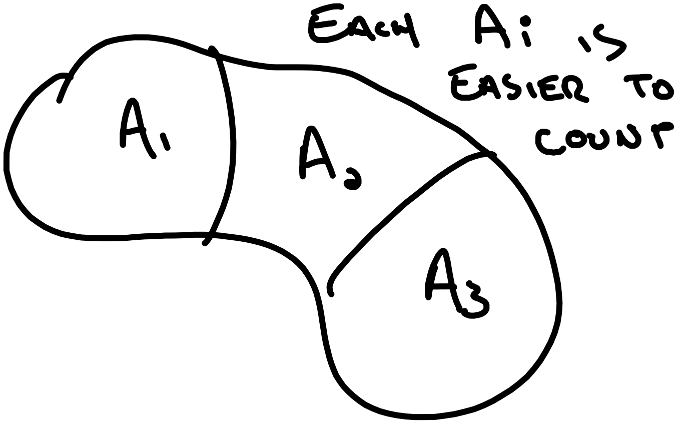
every item in A is in, at least, one subset A_i

union of all subsets A_i is the set A

$A_1 \cup A_2 \cup A_3 \cup \ldots = A$

## Count-by-partition:




Approach:
- partition desired set into subsets
- count each subset, add their cardinalities (number of items) together

Common Errors: subsets don't partition the set A
- not counting an item of A (violates partition definition: every item of A in at least one subset)
- double counting an item of A (violates partition definition: every item of A in at most one subset)

EXAMPLE  HOW MANY PICTURES OF UP TO 6 PEOPLE FROM 20 POSSIBLE

11 17 4 2 3 9 $\leftarrow P(20,6)$

1 2 7 14 3 $\leftarrow P(20,5)$

3 9 2 11 $\leftarrow P(20,4)$

1 2 3 $\leftarrow P(20,3)$

4 11 $\leftarrow P(20,2)$

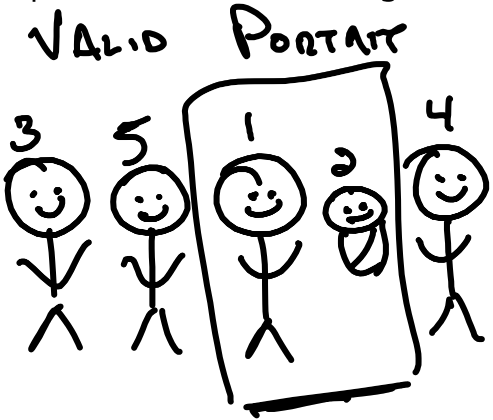
17 $\leftarrow P(20,1)$

Counting "moves":

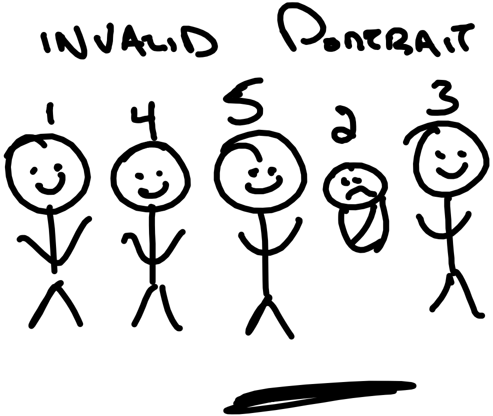
- count by complement
- count by partition
- count by simplify

## Count-by-simplification:

How many ways can we order 5 family members for a portrait if person 2 is a baby and must be on person 1's immediate right?




4!



Counting "moves":

- count by complement
- count by partition
- count by simplify

Mea Culpa:

"Count-by-simplification" isn't really a particular approach like others ...

point is, be on the lookout for equivilent problems more easily counted

## In Class Activity

How many passwords of length 10, made of lowercase characters, don't start with "qwerty"?
(hint: complement)

How many ways are there to order 3 people in a wedding photo for romeo and juliet?
Assume:

- there are 10 Montague's (Romeo's family, excluding him) who could be in the photo
- there are 7 Capulets (Juliet's family, excluding her) who could be in the photo
- Romeo and Juliet are too busy dancing to be in any picture
- Montagues and Capulets won't get in the same photo (that whole Tybalt / Mercutio thing...)

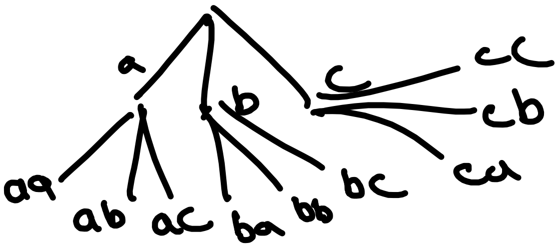
(hint: partition, simplify a bit)

How many ways are there to order 5 of 7 people in a family portrait such that person 1, if included, is not immediately to right of person 2?
(hint: partition, complement)

How many passwords of length 10, made of lowercase characters, don't start with "qwerty"?
(hint: complement)

$P(n,k)$

PASSWORDS LENGTH 2 from $\{a,b,c\}$



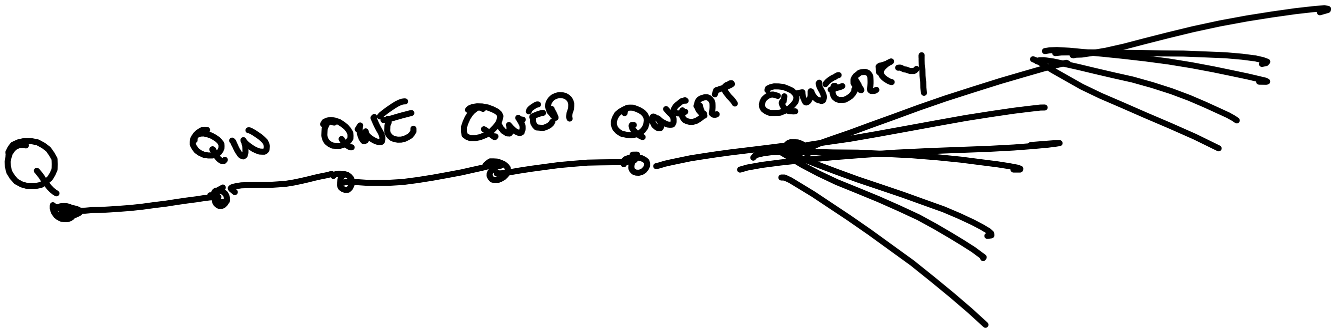

$3^2$

$26^{10}$ = # PASSWORDS LENGTH 10

$26^{10} - 26^4$ = # PASSWORDS LENGTH 10 DON'T START QWER

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Q | W | E | R | T | Y | _ | _ | _ | _ |
| 1 | 1 | 1 | 1 | 1 | 1 | 26 | 26 | 26 | 26 |

$$\underline{26^4}$$



How many ways are there to order 3 people in a wedding photo for romeo and juliet?
Assume:

- there are 10 Montague's (Romeo's family, excluding him) who could be in the photo
- there are 7 Capulets (Juliet's family, excluding her) who could be in the photo
- Romeo and Juliet are too busy dancing to be in any picture
- Montagues and Capulets won't get in the same photo (that whole Tybalt / Mercutio thing...)

(hint: partition, simplify a bit)

$$|P| = |P_C| + |P_M|$$

$$= P(7,3) + P(10,3)$$

How many ways are there to order 5 of 7 people in a family portrait such that person 1, if included, is not immediately to right of person 2?
(hint: partition, complement)

$P(7,5) - $ # WAYS PERSON 1 IS INCLUDED AND IMMEDIATE RIGHT OF 2

HOW MANY WAYS ORDER 5 OF 7 PEOPLE

$P(5,3)$ 2 1 _ _ _

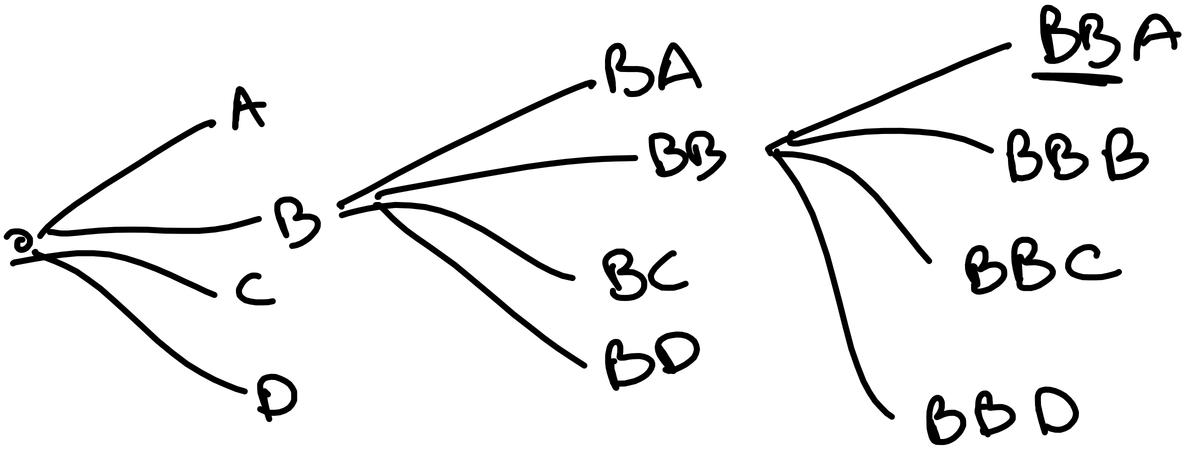
$P(5,3)$ _ 2 1 _ _

_ _ _ 2 1 _

_ _ _ _ 2 1

$$P(7,5) - 4 \cdot P(5,3)$$

PASSWORDS OF
LENGTH 3
FROM A, B, C, D




$16^3$