Professor Hamlin
Day 17

Agenda

1) Admin

HW 7 released - shorter and only 80 pts

Exam 2 next Friday in class

- practice problems released → try doing them before looking at the answer.

2) Exam Content
3) Summation notation
4) Strong induction
5) Induction: inequality

updated the practice problems just now, 1PM on Nov 8, to include them).

- **Bayes Rule will not be given on exam2, you should memorize this (worth your while down the road)**
- Binomial / Poisson distribution formulas won't be needed
- an image of the equations is included below

$E[x] = \sum_x x * P(X = x)$

$\texttt{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$

	No Repeat Selections	Repeat Selections
Order Matters	Permutations: $P(n,k) = \frac{n!}{(n-k)!}$	Product Rule: n^k
Order Doesn't Matter	Combinations: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$	Stars and Bars (Partitions of k identical objects into n groups): $\binom{k+n-1}{n-1}$

Exam Contents

→ Counting style probability e.g $\frac{|E|}{|S|}$

→ Expected Value / Variance

→ Bayes rule problem

→ BFS / DFS

→ Dijkstra algorithm

→ induction (equality or inequality)

Review: induction

Show: $x^0 + x^1 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}$

Base case: $n = 0$

$x^0 = \frac{1 - x^{0+1}}{1 - x}$

$1 = 1$ ✓

Conditional

$P \to Q$

WLOG: without loss of generality

Inductive Step: if $\underbrace{\text{statement } n}_{P}$, then $\underbrace{\text{statement } n+1}_{Q}$

P: $x^0 + x^1 + \dots + x^n = \frac{1-x^{n+1}}{1-x}$

Q: $x^0 + x^1 + \dots + x^n + x^{n+1} = \frac{1-x^{(n+1)+1}}{1-x}$

Always start w/ LHS of Q

Proof: Assume P holds (I.H.)

$$x^0 + x^1 + \dots + x^n + x^{n+1} = \underbrace{x^0 + x^1 + \dots + x^n} + x^{n+1}$$

$$= \frac{1-x^{n+1}}{1-x} + x^{n+1} \cdot \frac{(1-x)}{(1-x)}$$

$$= \frac{1-x^{n+1} + x^{n+1} - x \cdot x^{n+1}}{1-x}$$

$$= \frac{1-x^{(n+1)+1}}{1-x}$$

Summation Notation

We've been writing $1 + \dots + n$ → want quicker way: Summation!

$$1 + 2 + 4 + 8 + 16 + 32 + 64$$

$$= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$$

7 terms

$= \sum_{K=0}^{6} 2^K$ → by putting each value of K into this term, then sum them

6 last value of K

sigma

K=0 starting value of K

"The sum of 2^K where K goes from 0 to 6"

Exercise Express the following in sum notation

1) $9 + \overbrace{11 + 13}^{2} + 15 + 17$

$\sum_{K=1}^{5} 2K+7$ or $\sum_{K=0}^{4} 2K+9$

$\sum_{K=9}^{13} 2K-9$

$\boxed{\sum_{K=9}^{17} K+2 = \underset{K=9}{11} + \underset{K=10}{12} + 13 \ldots + 19}$

2) $1+2+3+4+5+\ldots+n$

$\sum_{K=1}^{n} K$ ~or~ $\sum_{K=0}^{n-1} K+1$

Compute each sum below:

1) $\sum_{K=10}^{12} 2K$

$\underset{K=10}{20} + \underset{K=11}{22} + \underset{K=12}{24}$

$\boxed{66}$

2) $\sum_{K=0}^{101} (-1)^K$

pattern

0 → 101

$\boxed{102 \text{ terms}}$

$-1^0 + -1^1 + -1^2 + -1^3 + \ldots + -1^{100} + -1^{101}$

$\underbrace{\underset{K=0}{1} + \underset{K=1}{-1}} + \underbrace{\underset{K=2}{1} + \underset{K=3}{-1}} + \underbrace{\underset{K=100}{1} + \underset{K=101}{-1}}$

$\boxed{0}$

Using summation in induction

from last class: Show $1+2+\dots+n = \frac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

then $1+2+\dots+n+n+1 = \frac{(n+1)(n+2)}{2}$

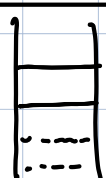
$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

starting w/ $\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + n+1$

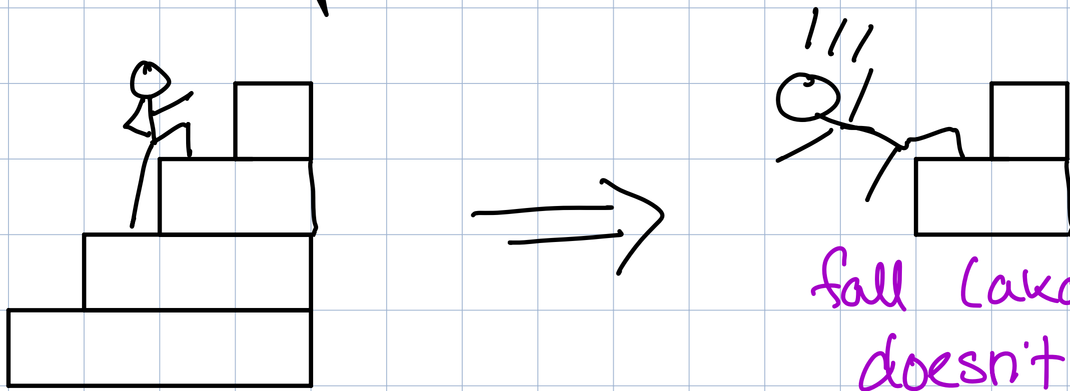
$\boxed{\text{Can always expand summation notation}}$

Back to induction

So far we've only needed to assume the previous step held



on a ladder, only need to assume previous rung on stairs what happens if I don't assume previous steps exist?



fall (aka proof doesn't go through)

weak: if statement n, then statement n+1

strong: if statemen 1, 2, ... n, then statement n+1

Example: Using only 3¢ and 4¢ coins can make any change starting at 6¢

10¢ = 2·3¢ + 4¢   19¢ = 4·4¢ + 1·3¢

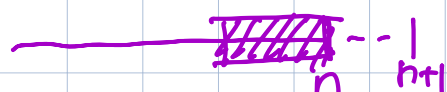
Base Cases: (more than 1!)

should have more base cases n=14

n=6	3¢ + 3¢
n=7	3¢ + 4¢
n=8	4¢ + 4¢

previously assumed just n!

Induction Step: if statement 6, 7, ... to n are true statement n+1 is true



Assume some combo of 3¢ & 4¢ produce n=6 ... n

Show for n+1:

n-3 can be represented by coins

Case 1: n's sum includes a 3¢ coin → switch out to 4¢ to make n+1

Case 2: n's sum only 4¢ coins →

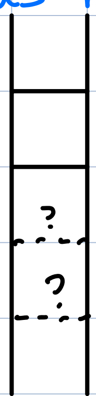
n-8 rep by coins

must be at least 2, 4¢ (by base case)
replace 2 4¢ w/ 3 3¢ for n+1

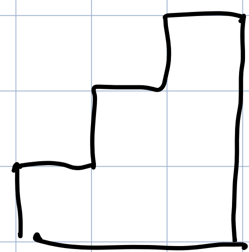
Strong Induction

Like weak induction, prove S(1), S(2) ....
but we use stronger assumption

Weak → just assume previous rung exists

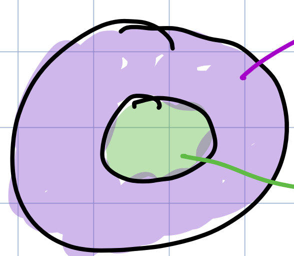


Strong → assume every step before exists



<u>Process:</u> 1) Prove statement for 1st few n
2) Show that S(1), S(2), ... S(n) implies S(n+1)

When should you use weak vs. strong?



→ set of problems can prove w/ strong induction

→ set of problems can prove w/ weak induction

Can always use strong, though if can use weak might be a simplier proof.

Exercise | Suppose two algorithms compute same task but take a different # of steps to do so

Algorithm 1: on input of size $n \to 2^N$ steps
Algorithm 2: on input of size $n \to N!$ steps

input size	1	2	3	4	5	6
Algo 1	2	4	8	16	32	64
Algo 2	1	2	6	24	120	720

1) Complete the table
2) Which algo for n=2 − 2
3) Which algo for n=5 − 1
4) Which algo for any size n? − 1
it goes faster

(weak) Induction w/ inequalities

"Prove that $2^N < N!$ for all N above 4"

Recall

Induction Recipe (from previous lesson)
1. define & remind: statement n
2. choose base case n & show it
3. write "inductive step: if S(n) then S(n+1)
4. Prove inductive step:
   a. assume statement n (inductive hypothesis)
   b. write statement n + 1 in two halves
      (tip: start at sum side, work to other side)
   c. apply assumption to get from one half to other

Statement: $2^N < N!$

Base case: $N = 4$

$2^4 < 4! \Rightarrow 16 < 24$ ✓

Inductive step if $\underbrace{\text{statement } n}_{P}$ then $\underbrace{n+1}_{Q}$

P: $2^N < N!$

Q: $\boxed{2^{N+1}} < \boxed{(N+1)!}$

$2^3 = 2 \cdot 2 \cdot 2$

$2 \cdot 2^2$

Proof: Assume P (I.H.)

starting with 2^{N+1}

$2^N < N!$

$2^{N+1} = 2 \cdot 2^N$

$2^{N+1} < 2 \cdot N!$ (by I.H.)

$< (N+1)N!$ (by $2 < N+1$ due to base case)

$< \boxed{(N+1)!}$

$2 < N+1$ if $N > 4$

$2 \cdot N! < (N+1)N! \rightarrow 2^{N+1} < 2 \cdot N! < (N+1)N!$

So this was some handwaving right? Sort of there are certain things we can do w/ inequalities we can't do w/ equalities

Move 1 add same thing to both sides, doesn't change inequality

if $3 < 4$ then $3+10 < 4+10$

$x < y \rightarrow x + c < y + c \quad \forall c \in \mathbb{R}$

Move 2 multiply by positive value, doesn't change inequality

if $3<4$ then $3\cdot 10 < 4\cdot 10$
if $x<y$ then $x\cdot c < y\cdot c \;\; \forall\, c\in\mathbb{R} \wedge c>0$

Move 3: multiply by negative value, swaps inequality direction

if $3<4$ then $3\cdot -1 > 4\cdot -1$
if $x<y$ then $xc > yc \;\; \forall\, x,y,c \in \mathbb{R}$ w/ $c<0$

Move 4: sum two inequalities (large side together & small side together)

if $3<4$ AND $5<6$ then $3+5 < 4+6$

if $x<y$ AND $w<z$ then $x+w < y+z$

Move 5: can replace smaller side of inequality w/ something smaller (or larger side w/ larger) if $3<7$ and $1<3$ then $1<7$

if $y<z$ and $x<y$ then $x<y$

Hint: One of most common manipulations for induction

Exercise) Complete table

	1	2	3	4	5	6
N^2	1	4	9	16	25	36
$N+10$	11	12	13	14	15	16

Show $N+10 < N^2$ for all N above some value

Proof) Base Case: $N=4$

$$4+10 < 4^2$$

$$14 < 16^2 \quad \checkmark$$

Inductive step if $\underbrace{\text{statement } n}_{P}$ then $\underbrace{n+1}_{Q}$

P: $N+10 < N^2$

Q: $(N+1)+10 < (N+1)^2$

Assume P

$$(N+1)+10 = (N+10)+1$$

$$< N^2+1$$

$0 < 2N$

for all value $N>4$

$N^2+1 < N^2+2N+1$

$< N^2+2N+1$
$< (N+1)^2$

ergo if
$N+1! < N^2+1$ then
$N+1! < N^2+2N+1$