1 Induction Example (Recipe & Rubric)

Our purpose here is to emphasize a recipe for how to approach writing an induction proof. On first look, it may appear a bit strict in requiring that certain steps be shown. In previous semesters, I've been a bit more relaxed with the formatting requirements and I've found that many students can lose track of precisely where they are in the proof, causing confusion. I hope that this recipe & rubric[1] help structure everyone's induction proofs so they their studies are more productive.

The green rubric boxes are immediately below the portion of the solution they refer to.

2 Geometric Series

Using induction, prove the following geometric series formula:

Let r be a real number not equal to zero. Then, for any natural number n greater than or equal to 1:

$$\sum_{i=1}^{n} a_1 r^{i-1} = a_1 \frac{1-r^n}{1-r}$$

Solution

Statement n is $\sum_{i=1}^{n} a_1 r^{i-1} = a_1 \frac{1-r^n}{1-r}$

Rubric

1 point: clearly writing what statement n is
(Students sometimes lose track of what they're proving, especially when writing induction proofs seperately from the problem statement. Labelling this right up top is helpful.)

Base Case (statement 1):

$$\sum_{i=1}^{1} ar^{i-1} = a_1 = a_1 \frac{1-r}{1-r}$$

Rubric

1 point: providing a clear base case index (e.g. here n=1) and writing what statement n=1 is
1 point: demonstration that the base case is true.

[1] Please note that the rubric point values here are suggestions only, different problems may weight things differently. We include them to give a rough sense of what that future rubric will be.

Inductive Step: if statement n then statement n + 1

Rubric

1 point: writing this label for the inductive step, you're welcome to write it exactly as "Inductive Step: if statement n then statement n + 1"
Feel free to write $S(n)$ instead of statement n if you prefer, though I worry that this more algebraic notation allows students to forget that its just a statement, you couldn't multiply it by 2 to get $2S(n)$.
(These are easy points to earn, and writing this title is a helpful reminder to reader and author alike: what follows is a proof of $S(n) \to S(n+1)$.)

Assume statement n is true, that is:

$$\sum_{i=1}^{n} a_1 r^{i-1} = a_1 \frac{1-r^n}{1-r}$$

Rubric

1 point: writing out statement n (the inductive hypothesis), explicitly.

Then:

$$\begin{aligned}
\sum_{i=1}^{n+1} a_1 r^{i-1} &= \sum_{i=1}^{n} a_1 r^{i-1} + a_1 r^n \\
&= a_1 \frac{1-r^n}{1-r} + a_1 r^n \\
&= a_1 \frac{r^n * (1-r) + 1 - r^n}{1-r} \\
&= a_1 \frac{r^n - r^{n+1} + 1 - r^n}{1-r} \\
&= a_1 \frac{1-r^{n+1}}{1-r}
\end{aligned}$$

Rubric

- 1 point: correctly writing the half of statement n + 1 (e.g. $\sum_{i=1}^{n+1} a_1 r^{i-1}$ above)
- 1 point: correctly writing the other half of statement n + 1 (e.g. $a_1 \frac{1-r^{n+1}}{1-r}$)
- 2 point: applying the inductive hypothesis (statement n) correctly within the reasoning
- 2 point: reasoning / algebra "glue" to get between either side of statement n + 1

Induction Tip

This last part is often the most challenging for students. Sometimes its the algebra which is tough and other times students are attempting to prove something which isn't true because they've made a mistake in the induction structure! Here's a few tips on setting up the induction structure so you can isolate your algebra challenges properly:

- Very often I find problems are easier to think about when we start[a] on the summation side of statement n + 1, $\sum_{i=1}^{n+1} a_1 r^{i-1}$, and work our way towards the other, simpler side of things.
- Towards the bottom of your page, write the second (simpler) side of things (i.e. $a_1 \frac{1-r^{n+1}}{1-r}$). Its worth a point and serves to remind us where we're headed.
- If you've got a summation to work from, try popping out that final term in the summation to set up applying our indutive hypothesis (statement n). Here's a silly little summation notation reminder of how that works:

$$\sum_{k=1}^{n+1} k = 1 + 2 + 3 + 4 + \ldots + n + (n+1) = (\sum_{k=1}^{n} k) + (n+1)$$

- Your reasoning must should[b] the inductive hypothesis (statement n), be on the lookout for a place to apply that assumption!

Following these four tips above yields the equalities below:

$$\begin{aligned} \sum_{i=1}^{n+1} a_1 r^{i-1} &= \sum_{i=1}^{n} a_1 r^{i-1} + a_1 r^n \\ &= a_1 \frac{1-r^n}{1-r} + a_1 r^n \\ \ldots \\ &= a_1 \frac{1-r^{n+1}}{1-r} \end{aligned}$$

Now that we've settled our induction structure, you can focus on the algebra required to fill in the ... above. (And, if the worst comes to it, know that you've scored the majority of the points on this induction proof ... students who struggle to connect the dots here might have an algebra challenge but their induction skills are solid).

[a] To be clear, you can right a correct induction proof, earning full credit, starting from either side of an equality / inequality. Working from complex (summation) to simple often helps students though.

[b] Should you not use it, then you've got a proof of statement n + 1 which doesn't rely on statement n. It may be a valid proof but its not an induction proof!