Professor Hamlin
Day 8

Agenda
1) Admin → regrades → formatting penalties on HW1

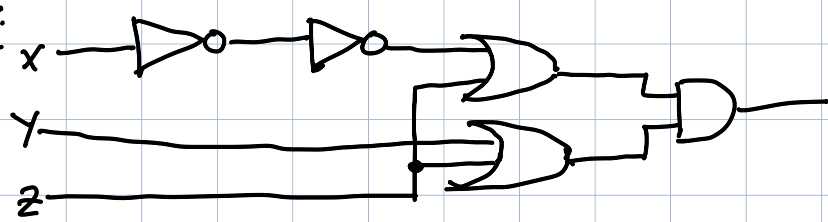
20% HW1 10% HW3 1/2
30% HW1 10% HW3 1/3

2) Review
3) Pigeonhole Principle
4) Counting
→ Product rule → cartesian product
→ sum rule → inclusion & exclusion

Review:
sets: represent as bit strings
logic & set operation AND / INTersection

algebra: simplifying expression
Circuits: wire, gates: AND, OR, NOT, XOR

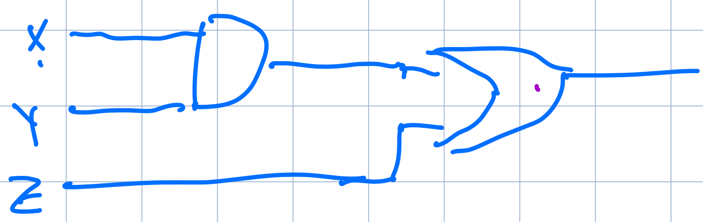
Exercise:



1) write boolean expression
$(\neg\neg x \vee z) \wedge (y \vee z)$

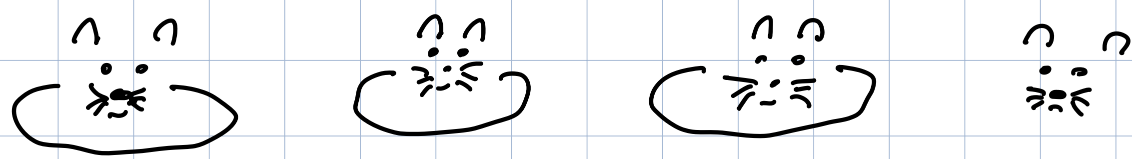
2) simplify expression
$(x \vee z) \wedge (y \vee z)$ double negation
$z \vee (x \wedge y)$ dist.

3) redraw circuit of simplified expression



Pigeonhole principle

(imagine) I have 3 cat beds and 4 cats



How can cats be distributed on the beds?

B1 C1
B2 C2 C4
B3 C3

~ or ~

B1 C1 C2 C3 C4
B2
B3

At least one bed is going to have at least two cats!

But we don't know all beds will have a cat or the exact # of cats

We can say if we have 4 (N) cats and 3 (K) beds then at least one bed will have at minimum $\lceil 4/3 \rceil$ cats

$\lceil N/K \rceil$ ← ceiling, when dividing always round up

$\lceil 6/4 \rceil = \lceil 1.5 \rceil = 2$ $\lfloor 6/4 \rfloor = \lfloor 1.5 \rfloor = 1$

$\lceil 5/4 \rceil = \lceil 1.25 \rceil = 2$

Let's try this out ~ 3 beds $(N=3)$

$(k=)$ Number of cats	0	1	2	3	4	5	6	7	8	9	10	11
$\left(\lceil N/k \rceil\right)$ guarenteed minimum of cats on one bed	–	1 $\lceil 1/3 \rceil$	1 $\lceil 2/3 \rceil$	1	2 $\lceil 4/3 \rceil$	2	2	3 $\lceil 7/3 \rceil$	3	3	4 $\lceil 10/3 \rceil$	4

B1	B2	B3
C1	C2	C3/C4

Pigeonhole Principle if we divide N items into K piles there exists a pile with a minimum of $\lceil N/K \rceil$ items

"$\exists$ pile w/ at least $\lceil N/K \rceil$ items" items

minimum (items) # of cats in at least one bed (pile)

$$X = \left\lceil \frac{N}{K} \right\rceil$$

N → # of cats

K → # of beds piles

Exercise

1) We have 3 pigeons and 2 nests. How many pigeons, at minimum, will be on at least one nest? $N=3$ $K=2$

$$\left\lceil 3/2 \right\rceil = \boxed{2}$$

2) If we group people in class by birth month how many people will be in the largest group at minimum? $N=149$ $K=12$

$$\left\lceil 149/12 \right\rceil = \boxed{13}$$

Say we want to publish exam grades online.

- Each student has a 2-digit secret hex code
- If you know it, can look up grade
- otherwise anonymous

Code	C1	DF	19	9F	B2	24	
Grade	A-	A	B+	B	A	A-	....

How many students can we support w/ 2-digits of hex? (We don't want collisions!)

N = # of students

k = # of hex codes

i.e. $1 = \lceil N/k \rceil$

What is k? $\underbrace{0\text{-}F}_{16}$ $\underbrace{0\text{-}F}_{16}$

$\boxed{256}$

$FF = 15 \cdot 16^1 + 15 \to 16^2 - 1$

smallest 0

largest 16^2-1

16^2 256

0 → 16^2-1

So N can't be any bigger than that!

$\left\lceil \frac{257}{256} \right\rceil = 2$

For a class of 800 we will need more Hex digits!

How many? — Counting

Counting

A computer can try and guess a password 1000 times a second* how will a 4 digit passcode hold up

$\boxed{9\ 5\ 7\ 2}$ $\boxed{1\ 5\ 8\ 3}$

- 4 numbers? $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ $\boxed{10s}$
- 4 letters? $26 \cdot 26 \cdot 26 \cdot 26 = 26^4$ $\boxed{457s}$

What about longer

- 8 numbers? $10 \cdot 10 \ldots = 10^8$ $\boxed{10^5 s}$
- 8 letters? $26 \cdot 26 \cdot 26 \ldots = 26^8$ $\boxed{2.1 \times 10^9 s}$

$\boxed{\text{* closer to 14 billion per second}}$

Longer passwords are better than ones w/ more characters!

Product rule

Getting dressed in the morning

shirt  3 shirts

~ and ~

pants  2 pants

~ and ~

socks  2 pairs of socks

How many different outfits can I wear?

Shirts $= \{A, B, C\}$
pants $= \{1, 2\}$
Socks $= \{a, b\}$

$(A, 1, a)$ $(B, 2, b)$
$(A, 1, b)$ $\vdots$
$\vdots$
$(C, 2, b)$

We can capture this formally - cartesian product
set of elements in A paired w/ all of elements in B

$A = \{1, 2, 3\}$ $B = \{1, 2\}$

$A \times B = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$

Tuple: may repeat, order matters
$(1,2) \neq (2,1)$

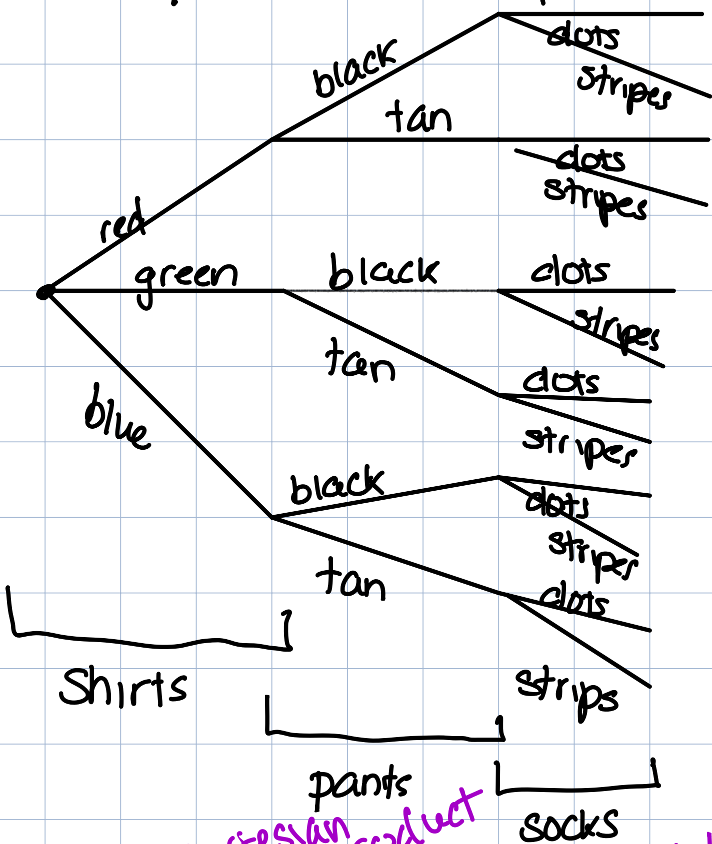
Note $A \times B \neq B \times A \rightarrow \{(1,1)(1,2), (1,3)\ldots$

Exercise: What is cartesian product of
shirts $= \{\overset{R}{\text{red}}, \overset{B}{\text{blue}}\}$ pants $= \{\overset{B}{\text{black}}, \overset{T}{\text{tan}}\}$
socks $= \{\overset{D}{\text{dots}}, \overset{S}{\text{stripes}}\}$

shirts $\times$ pants $\times$ socks $= \{(R, B, D), (R, B, S),$
$(R, T, D), (R, T, S),$
$(B, B, D), (B, B, S),$
$(B, T, D), (B, T, S)\}$

$\boxed{\text{8 options!}}$

How many w/ 3 shirt options? {red, green, blue}



shirts x pants x socks has...

$3 \times 2 \times \cancel{2}\,4$

$\boxed{\text{12 outfits}}$

cartesian product cardinality times

Formally: $|A \times B| = |A| * |B|$

$|A \times B \times C| = |A| * |B| * |C|$

$\vdots$

Product rule is the cardinality of cartesian product of the set options

We use product rule when we have an 'and' between choices.

shirt AND pants AND socks

not

shirt OR dress

Exercise | 1) How many passwords of length 4 can be made w/ 4 lower case letters?

L = set of lower case letters

$|L \times L \times L \times L| = |L| * |L| * |L| * |L|$

$26 * 26 * 26 * 26 = \boxed{26^4}$

2) ... upper & lowercase letters

A = set of upper/lowercase letters

$|A \times A \times A \times A| = 52 \cdot 52 \cdot 52 \cdot 52$

$\boxed{52^4}$

3) ... if 1st letter is 'a' and the rest are lowercase?

$|\{a\} \times L \times L \times L| = |\{a\}| \times 26 \times 26 \times 26$

$1 \times 26^3 = \boxed{26^3}$

4) if 1st letter is 'a', 'b', or 'c', and rest are lower case?

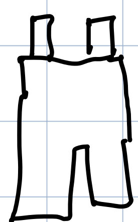
$S = \{a, b, c\}$

$|S \times L \times L \times L| = \boxed{3 \cdot 26^3}$

$|S \times S \times L \times L|$

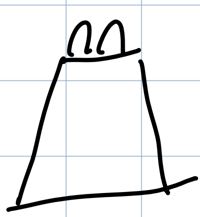
Sum Rule

Getting dressed - overalls or dress



overalls = {denim, corderoy}

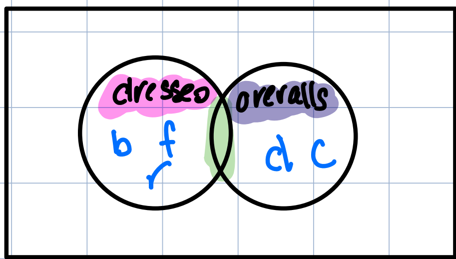
~or~



dress = {blue, flowers, red}

How many options? $\boxed{5}$

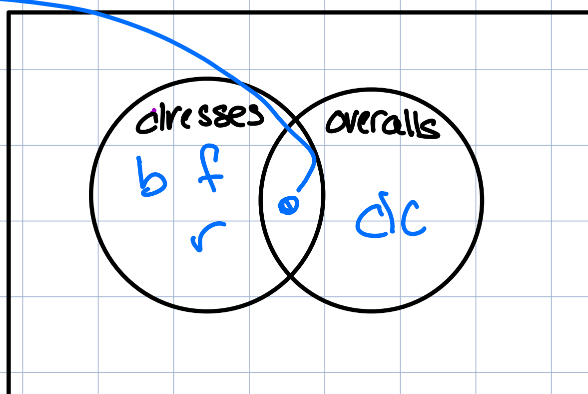
Formally if the sets A & B are disjoint (no overlap) the items in $|A \cup B| = |A + |B|$



$$|A \cup B| = \underbrace{|A|}_{3} + \underbrace{|B|}_{2}$$

$$\boxed{5}$$

But what about .... the overall dress??

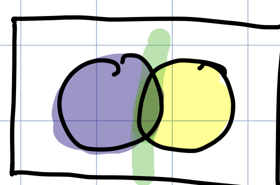


$$|A \cup B| \neq |A| + |B|$$
$$\neq 4 + 3$$
$$6 \neq 7$$

$\boxed{A \cap B \text{ is counted twice!}}$

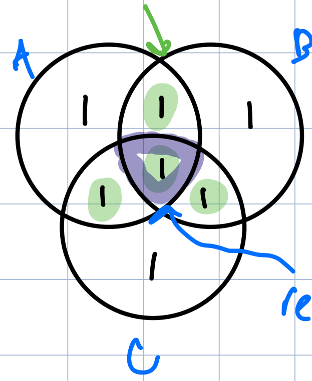
Principle of Inclusion-exclusion (PIE): when counting union # items in A + items in B minus any in the intersection double counted

$$|A \cup B| = |A| + |B| - |A \cap B|$$



PIE for 3 sets:

double counted



removed 3 times!

$$|A \cup B \cup C| = |A| + |B| + |C| - \underline{|A \cap B|} - \underline{|A \cap C|} - \underline{|B \cap C|} + \underline{|A \cap B \cap C|}$$

$|M \cup S \cup C| = 17$

Example: 17 employees w/ 3 different roles (manage, stock, checkout) — M, S, C

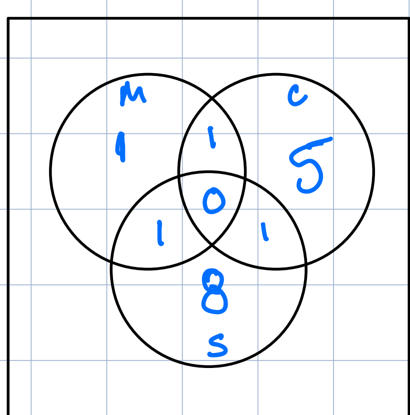
- 3 trained as managers $|M| = 3$
- 10 trained to stock shoes $|S| = 10$
- 7 trained at checkout $|C| = 7$
- 1 employee has double training in each pair of jobs $|M \cap S| = |M \cap C| = |S \cap C| = 1$

How many employees are trained to do all three jobs? $|M \cap S \cap C| = ?$

$$|M \cup C \cup S| = |M| + |C| + |S| - |M \cap S| - |M \cap C| - |S \cap C| + |M \cap S \cap C|$$

$$17 = 3 + 10 + 7 - 1 - 1 - 1 + x$$

$$x = 0$$



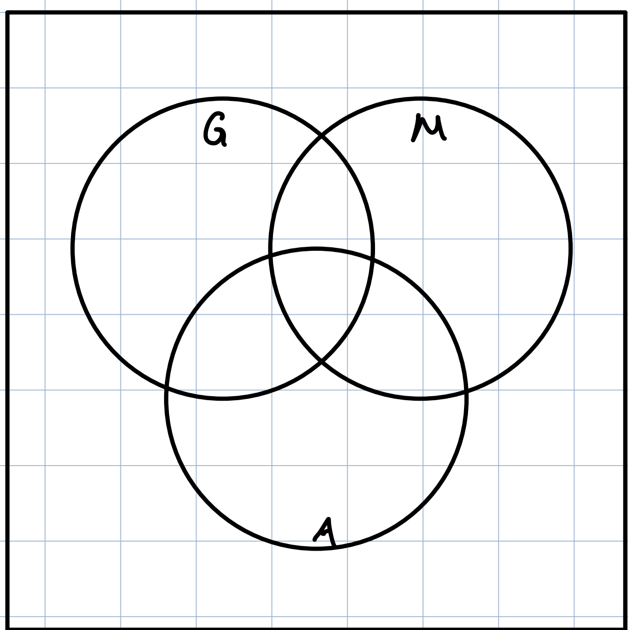
Exercise:

Of the 196 kindergarden students which like gym or music or art:

45 like gym class
90 like music class
100 like art class
20 like both gym and music
13 like both gym and art
7 like both art and music

1) how many students like gym or music?
2) how many students like all 3 subjects?
~~3) how many students like gym but nothing else~~

$|G \cup M \cup A| = 196$
$|G| = 45$
$|M| = 90$
$|A| = 100$
$|G \cap M| = 20$
$|G \cap A| = 13$
$|M \cap A| = 7$



1) $|G \cup M| = |G| + |M| - |G \cap M|$
$= 45 + 90 - 20$
$= 115$

2) $|G \cap M \cap A| = ?$
$196 = |G| + |M| + |A| - |G \cap M| - |G \cap A| - |M \cap A| + |G \cap M \cap A|$

$196 = 45 + 90 + 100 - 20 - 13 - 7 + x$
$x = 1$